Mth 205 Exercises 2.5, 2.6 Zill 5th

2.5 #5 $\frac{dy}{dx} + y = e^{3x}$

$\mu(x) = e^{\int dx} = e^{x}$

$e^{x}\frac{dy}{dx} + e^{x} y = e^{4x}$

$\frac{d}{dx}[e^{x}y] = e^{4x}$

$e^{x}y = \int e^{4x}\,dx$

$e^{x}y = \frac{1}{4}e^{4x} + c$

$y = \frac{1}{4}e^{3x} + ce^{-x}$ $\qquad -\infty < x < \infty$

2.5 #7 $y' + 3x^2 y = x^2$

$\mu(x) = e^{\int 3x^2 dx} = e^{x^3}$

$e^{x^3}y' + 3x^2 e^{x^3} y = e^{x^3}x^2$

$\frac{d}{dx}[y\,e^{x^3}] = e^{x^3}x^2$

$y\,e^{x^3} = \int x^2 e^{x^3}\,dx$

$= \frac{1}{3}e^{x^3} + c$

$y = \frac{1}{3} + ce^{-x^3}$

2.5#9

$$x^2y' + xy = 1$$

$$\frac{x^2y'}{x^2} + \frac{xy}{x^2} = \frac{1}{x^2}$$

$$y' + \frac{1}{x}y = \frac{1}{x^2}$$

$$\mu(x) = e^{\int \frac{1}{x}dx} = e^{\ln x} = x \qquad (x>0)$$

$$xy' + y = \frac{1}{x}$$

$$\frac{d}{dx}[xy] = \frac{1}{x}$$

$$xy = \int \frac{1}{x}dx$$

$$xy = \ln x + c$$

$$y = \frac{1}{x}\ln x + \frac{c}{x} \qquad (x>0)$$

2.5#13

$$x\,dy = (x\sin x - y)dx$$

$$\frac{dy}{dx} + \frac{y}{x} = \sin x$$

$$\mu(x) = e^{\int \frac{1}{x}dx} = e^{\ln x} = x \qquad (x>0)$$

$$x\frac{dy}{dx} + y = x\sin x$$

$$\frac{d}{dx}[xy] = x\sin x$$

$$xy = \int x\sin x\,dx$$

$u = x \quad dv = \sin x\,dx$
$du = dx \quad v = -\cos x$
$-x\cos x + \int \cos x\,dx$

$$xy = \sin x - x\cos x + c$$

$$y = \frac{1}{x}\sin x - \cos x + \frac{c}{x} \qquad x>0$$

2.5 #15 $(1+e^x)\frac{dy}{dx}+e^x y=0$

$$\frac{dy}{dx}+\frac{e^x}{1+e^x}y=0$$

$$\mu(x)=e^{\int\frac{e^x}{1+e^x}dx}=e^{\ln(1+e^x)}=1+e^x$$

$$(1+e^x)\frac{dy}{dx}+e^x y=0$$

$$\frac{d}{dx}[(1+e^x)y]=0$$

$$(1+e^x)y=c$$

$$y=\frac{c}{1+e^x} \qquad -\infty<x<\infty$$

2.5 #17 $\cos x\frac{dy}{dx}+y\sin x=1$

$$\frac{dy}{dx}+\frac{\sin x}{\cos x}y=\frac{1}{\cos x}$$

$$\mu(x)=e^{\int\frac{\sin x}{\cos x}dx}=e^{-\ln|\cos x|}$$

$$=\frac{1}{|\cos x|}$$

$$=\frac{1}{\cos x}\quad\left(-\frac{\pi}{2}<x<\frac{\pi}{2}\right)$$

$$\frac{1}{\cos x}\frac{dy}{dx}+\frac{\sin x}{\cos^2 x}y=\frac{1}{\cos^2 x}$$

$$\frac{d}{dx}[y\sec x]=\frac{1}{\cos^2 x}$$

$$y\sec x=\int\sec^2 x\,dx$$

$$y\sec x=\tan x+c$$

$$y=\frac{\tan x}{\sec x}+\frac{c}{\sec x}$$

$$y=\sin x+c\cos x$$

2.5# 19 $\quad x\dfrac{dy}{dx}+4y = x^3 - x$

$$\frac{dy}{dx}+\frac{4}{x}y = x^2-1$$

$$\mu(x) = e^{\int \frac{4}{x}dx} = e^{4\ln x} = x^4 \qquad x>0$$

$$x^4\frac{dy}{dx}+4x^3y = x^6 - x^4$$

$$\frac{d}{dx}[x^4y] = x^6-x^4$$

$$\int d[x^4y] = \int (x^6-x^4)dx$$

$$x^4y = \frac{x^7}{7}-\frac{x^5}{5}+c$$

$$y = \frac{x^3}{7}-\frac{x}{5}+\frac{c}{x^4} \qquad 0<x$$

2.5#21 $\quad x^2y' + x(x+2)y = e^x$

$$y' + \frac{x(x+2)}{x^2}y = \frac{e^x}{x^2}$$

$$y' + \left(1+\frac{2}{x}\right)y = \frac{e^x}{x^2}$$

$$\mu(x) = e^{\int (1+\frac{2}{x})dx} = e^{x+2\ln x}$$

$$= x^2e^x \qquad x>0$$

$$x^2e^xy' + x^2e^x\left(1+\frac{2}{x}\right)y = e^{2x}$$

$$x^2e^xy' + e^x(x^2+2x)y = e^{2x}$$

$$\frac{d}{dx}[x^2e^xy] = e^{2x}$$

$$x^2e^xy = \int e^{2x}dx$$

$$x^2e^xy = \frac{1}{2}e^{2x}+c$$

$$y = \frac{e^x}{2x^2}+c\frac{e^{-x}}{x^2} \qquad x>0$$

2.5 #25

$$y\,dx + (xy + 2x - ye^{y})\,dy = 0$$

$$\frac{dx}{dy} + \left(x + \frac{2x}{y} - e^{y}\right) = 0$$

$$\frac{dx}{dy} + \left(1 + \frac{2}{y}\right)x = e^{y}$$

$$\mu(y) = e^{\int (1 + \frac{2}{y})dy} = e^{y + 2\ln y} \qquad y > 0$$

$$= y^2 e^{y}$$

$$y^2 e^{y}\frac{dx}{dy} + \left(1 + \frac{2}{y}\right)y^2 e^{y} x = y^2 e^{2y}$$

$$y^2 e^{y}\frac{dx}{dy} + (y^2 + 2y)e^{y} x = y^2 e^{2y}$$

$$\frac{d}{dy}\left[y^2 e^{y} x\right] = y^2 e^{2y}$$

$$y^2 e^{y} x = \int y^2 e^{2y}\,dy$$

$$y^2 e^{y} x = \frac{y^2 e^{2y}}{2} - \frac{y e^{2y}}{2} + \frac{1}{4}e^{2y} + c$$

$$x = \frac{e^{y}}{2} - \frac{e^{y}}{2y} + \frac{e^{y}}{4y^2} + \frac{c}{y^2 e^{y}}$$

$y > 0$

$u = y^2 \quad dv = e^{2y}$

$du = 2y\,dy \quad v = \frac{1}{2}e^{2y}$

$$\frac{y^2 e^{2y}}{2} - \int y e^{2y}\,dy$$

$u = y \quad dv = e^{2y}dy$

$du = dy \quad v = \frac{1}{2}e^{2y}$

$$\frac{y e^{2y}}{2} - \int \frac{1}{2}e^{2y}\,dy$$

$$\frac{y e^{2y}}{2} - \frac{1}{4}e^{2y}$$

2.5 #31 $\frac{dy}{dx} + y = \frac{1-e^{-2x}}{e^x+e^{-x}}$

$$\mu(x) = e^{\int dx} = e^x$$

$$e^x \frac{dy}{dx} + e^x y = \frac{e^x(1-e^{-2x})}{e^x+e^{-x}}$$

$$\frac{d}{dx}[y e^x] = \frac{e^x - e^{-x}}{e^x+e^{-x}}$$

$$y e^x = \int \frac{e^x - e^{-x}}{e^x+e^{-x}}\, dx$$

$$= \int \frac{\sinh x}{\cosh x}\, dx$$

$$y e^x = \ln|\cosh x| + c$$

$\cosh x > 0$ all $x$

$$y = e^{-x} \ln(\cosh x) + Ce^{-x}$$

2.5 #39 $y' = (10-y)\cosh x$

$$\frac{dy}{dx} + (\cosh x) y = 10 \cosh x$$

$$\mu(x) = e^{\int \cosh x\, dx} = e^{\sinh x}$$

$$e^{\sinh x} \frac{dy}{dx} + \cosh x\, e^{\sinh x} y = 10 e^{\sinh x} \cosh x$$

$$\frac{d}{dx}\left[e^{\sinh x} y\right] = 10 e^{\sinh x} \cosh x$$

$$e^{\sinh x} y = 10 \int e^{\sinh x} \cosh x\, dx$$

$$e^{\sinh x} y = 10 e^{\sinh x} + c$$

$$y = 10 + Ce^{-\sinh x}$$

$-\infty < x < \infty$

2.5 #41 $\frac{dy}{dx} + 5y = 20 \qquad y(0) = 2$

$$\mu(x) = e^{\int 5\,dx} = e^{5x}$$

$$e^{5x}\frac{dy}{dx} + 5e^{5x}y = 20e^{5x}$$

$$\frac{d}{dx}\left[e^{5x}y\right] = 20e^{5x}$$

$$e^{5x}y = \int 20e^{5x}\,dx$$

$$e^{5x}y = 4e^{5x} + c$$

$$y = 4 + Ce^{-5x}$$

$$2 = 4 + c$$

$$-2 = c$$

$$y = 4 - 2e^{-5x} \qquad -\infty < x < \infty$$

2.5 #45 $y' + (\tan x)y = \cos^2 x \qquad y(0) = -1$

$$\mu(x) = e^{\int \tan x\,dx} = e^{-\ln \cos x} \qquad -\frac{\pi}{2} < x < \pi/2$$

$$= \frac{1}{\cos x}$$

$$\frac{1}{\cos x}y' + \frac{1}{\cos x}\tan x\, y = \frac{\cos^2 x}{\cos x}$$

$$\sec x\, y' + \sec x \tan x\, y = \cos x$$

$$\frac{d}{dx}\left[(\sec x)y\right] = \cos x$$

$$y \sec x = \int \cos x\,dx$$

$$y \sec x = \sin x + c$$

$$y = \sin x \cos x + c\cos x$$

$$-1 = c$$

$$y = \sin x \cos x - \cos x$$

25#47 $\quad \dfrac{dT}{dt} = k(T-50)$ ; $\quad k$ constant, $\quad T(0) = 200$

$$\frac{dT}{dt} - kT = -50k$$

$$u(t) = e^{\int -k\,dt} = e^{-kt}$$

$$e^{-kt}\frac{dT}{dt} - ke^{-kt}T = -50ke^{-kt}$$

$$\frac{d}{dt}\left[e^{-kt}T\right] = -50ke^{-kt}$$

$$e^{-kt}T = 50\int ke^{-kt}\,dt$$

$$e^{-kt}T = 50e^{-kt} + c$$

$$T = 50 + ce^{kt}$$

$$200 = 50 + c$$

$$150 = c$$

$$T = 50 + 150e^{kt}$$

2.5#53 $\quad \dfrac{dy}{dt} = \dfrac{y}{y-x}$ , $\quad y(5) = 2$

$$(y-x)\,dy = y\,dx$$

$$y - x = y\frac{dx}{dy}$$

$$1 - \frac{x}{y} = \frac{dx}{dy}$$

$$\frac{dx}{dy} + \frac{1}{y}x = 1$$

$$u(y) = e^{\int \frac{1}{y}dy} = e^{\ln y} = y$$

$$y\frac{dx}{dy} + x = y$$

$$\frac{d}{dy}[yx] = y$$

$$xy = \int y\,dy$$

$$xy = \frac{y^2}{2} + c$$

$$10 = \frac{4}{2} + c$$

$$8 = c$$

$$\boxed{x = \frac{y}{2} + \frac{8}{y}}$$

$y > 0$

2.5 #55 $\frac{dy}{dx} + 2y = f(x)$, $f(x) = \begin{cases} 1, & 0 \le x \le 3 \\ 0, & x > 3 \end{cases}$ $y(0)$

Solve Separately

$$\frac{dy}{dx} + 2y = 1$$

$$\mu(x) = e^{\int 2\,dx} = e^{2x}$$

$$e^{2x}\frac{dy}{dx} + 2e^{2x}y = e^{2x}$$

$$\frac{d}{dx}\left[e^{2x}y\right] = e^{2x}$$

$$e^{2x}y = \int e^{2x}\,dx$$

$$e^{2x}y = \frac{1}{2}e^{2x} + c$$

$$y(0)=0 \Rightarrow 0 = \frac{1}{2} + c$$

$$c = -\frac{1}{2}$$

$$y = \frac{1}{2} - \frac{1}{2}e^{-2x}$$

$$\frac{dy}{dx} + 2y = 0$$

$$e^{2x}\frac{dy}{dx} + 2e^{2x}y = 0$$

$$\frac{d}{dx}\left[e^{2x}y\right] = 0$$

$$e^{2x}y = C_1$$

$$y = C_1e^{-2x}$$

$$y = \begin{cases} \frac{1}{2} - \frac{1}{2}e^{-2x} & 0 \le x \le 3 \\ C_1e^{-2x} & x > 3 \end{cases}$$

$$y = \begin{cases} \frac{1}{2} - \frac{1}{2}e^{-2x} & 0 \le x \le \\ \frac{1}{2}e^{6-2x} - \frac{1}{2}e^{-2x} & x > \end{cases}$$

For Continuity $\lim_{x \to 3^+} C_1e^{-2x} = \frac{1}{2} - \frac{1}{2}e^{-6}$

$$C_1e^{-6} = \frac{1}{2} - \frac{1}{2}e^{-6}$$

$$C_1 = \frac{1}{2}e^{6} - \frac{1}{2}$$

2.5 # 57 $\quad \frac{dy}{dx} + 2xy = f(x), \quad f(x) = \begin{cases} x, & 0 \le x < 1 \\ 0, & x \geq 1 \end{cases} \quad y(0) = 2$

$$\frac{dy}{dx} + 2xy = x$$

$$\mu(x) = e^{\int 2x\,dx} = e^{x^2}$$

$$e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y = xe^{x^2}$$

$$\frac{d}{dx}\left[e^{x^2}y\right] = xe^{x^2}$$

$$e^{x^2}y = \int xe^{x^2}\,dx$$

$$e^{x^2}y = \frac{1}{2}e^{x^2} + c$$

$$y = \frac{1}{2} + ce^{-x^2}$$

$$2 = \frac{1}{2} + c$$

$$\frac{3}{2} = c$$

$$y = \frac{1}{2} + \frac{3}{2}e^{-x^2} \qquad 0 \le x < 1$$

$$\frac{dy}{dx} + 2xy = 0$$

$$\vdots$$

$$\frac{d}{dx}\left[e^{x^2}y\right] = 0$$

$$e^{x^2}y = c_1$$

$$y = c_1e^{-x^2}$$

$$y = \begin{cases} \frac{1}{2} + \frac{3}{2}e^{-x^2} & 0 \le x < 1 \\ c_1e^{-x^2} & x \geq 1 \end{cases}$$

$$y = \begin{cases} \frac{1}{2} + \frac{3}{2}e^{-x^2} & 0 \le x < 1 \\ \frac{1}{2}e^{1-x^2} + \frac{3}{2}e^{-x^2} & x \geq 1 \end{cases}$$

Continuity at $x=1 \Rightarrow \frac{1}{2} + \frac{3}{2e} = c_1e^{-1} \Rightarrow c_1 = \frac{1}{2}e + \frac{3}{2}$

2.6 Bernoulli's Equation

$$\frac{dy}{dx} + P(x)y = f(x)\,y^n$$

$n$ real

The substitution $w = y^{1-n}$ leads to a linear equation which can be solved accordingly.

2.6 #1

$$x\frac{dy}{dx} + y = \frac{1}{y^2}$$

$$\frac{dy}{dx} + \frac{1}{x}y = \frac{1}{x}y^{-2}$$

let $w = y^{1+2} = y^3 \Rightarrow y = w^{1/3}$, $y^{-2} = w^{-2/3}$

$$\frac{dw}{dx} = 3y^2\frac{dy}{dx}$$

$$\frac{1}{3y^2}\frac{dw}{dx} = \frac{dy}{dx}$$

$$\frac{1}{3y^2}\frac{dw}{dx} + \frac{1}{x}y = \frac{1}{x}y^{-2}$$

$$\frac{dw}{dx} + \frac{3}{x}y^3 = \frac{3}{x}$$

$$\frac{dw}{dx} + \frac{3}{x}w = \frac{3}{x}$$

$$\mu(x) = e^{\int\frac{3}{x}dx} = e^{3\ln x} = x^3$$

$$x^3\frac{dw}{dx} + 3x^2w = 3x^2$$

$$\frac{d}{dx}[x^3w] = 3x^2$$

$$x^3w = x^3 + c$$

$$x^3y^3 = x^3 + c$$

$$y^3 = 1 + \frac{c}{x^3}$$

2.6 #5

$$x^2\frac{dy}{dx}+y^2=xy$$

$$\frac{dy}{dx}+\frac{y^2}{x^2}=\frac{y}{x}$$

$$\frac{dy}{dx}-\frac{y}{x}=-\frac{y^2}{x^2}$$

let $w=y^{1-2}=y^{-1}$

$$\frac{dw}{dx}=-\frac{1}{y^2}\frac{dy}{dx}$$

$$\frac{dy}{dx}=-y^2\frac{dw}{dx}$$

$$-y^2\frac{dw}{dx}-\frac{y}{x}=-\frac{y^2}{x^2}$$

$$\frac{dw}{dx}+\frac{1}{yx}=\frac{1}{x^2}$$

$$\frac{dw}{dx}+\frac{1}{x}w=\frac{1}{x^2}$$

$$\mu(x)=e^{\int\frac{1}{x}dx}=e^{\ln x}=x \qquad x>0$$

$$x\frac{dw}{dx}+w=\frac{1}{x}$$

$$\frac{d}{dx}[xw]=\frac{1}{x}$$

$$xw=\int\frac{1}{x}dx$$

$$xw=\ln x+c$$

$$x\frac{1}{y}=\ln x+c$$

$$x=y\ln x+yc$$

or

$$\frac{1}{y}=\frac{\ln x}{x}+\frac{c}{x}$$

26 # 7 $\quad x^2\frac{dy}{dx} - 2xy = 3y^4 \qquad y(1) = \frac{1}{2}$

$$\frac{dy}{dx} - \frac{2y}{x} = \frac{3y^4}{x^2}$$

Let $w = y^{1-4} = y^{-3}$

$$\frac{dw}{dx} = -3y^{-4}\frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{y^4}{3}\frac{dw}{dx}$$

$$-\frac{1}{3}y^4\frac{dw}{dx} - \frac{2y}{x} = \frac{3y^4}{x^2}$$

$$\frac{dw}{dx} + \frac{6y^{-3}}{x} = \frac{-9}{x^2}$$

$$\frac{dw}{dx} + \frac{6}{x}w = \frac{-9}{x^2}$$

$$\mu(x) = e^{\int \frac{6}{x}dx} = e^{6\ln x} = x^6 \qquad x > 0$$

$$x^6\frac{dw}{dx} + 6x^5 w = -9x^4$$

$$\frac{d}{dx}[x^6 w] = -9x^4$$

$$x^6 w = \frac{-9x^5}{5} + c$$

$$x^6 y^{-3} = \frac{-9x^5}{5} + c$$

$$\frac{x^6}{y^3} = \frac{-9x^5}{5} + c$$

$$\frac{1}{\left(\frac{1}{2}\right)^3} = -\frac{9}{5} + c$$

$$8 + \frac{9}{5} = c$$

$$\frac{49}{5} = c$$

$$y^{-3} = -\frac{9}{5x} + \frac{49}{5x^6}$$