Mth 205

Solutions to 4.1 – 4.3 Zill 5th Selected exercises

4.1 #1 $y = c_1 e^{x} + c_2 e^{-x}$ is a solution on $(-\infty, \infty)$ of $y'' - y = 0$. Find a member satisfying $y(0) = 0$, $y'(0) = 1$

$y' = c_1 e^{x} - c_2 e^{-x}$

$y(0) = 0 = c_1 + c_2$

$c_1 = -c_2$

$y'(0) = 1 = c_1 - c_2$

$1 = -c_2 - c_2$

$-\frac{1}{2} = c_2 \Rightarrow c_1 = \frac{1}{2}$

$$\boxed{y = \frac{1}{2}e^{x} - \frac{1}{2}e^{-x}}$$

4.1 #7 Find two members of the family of solutions of $xy'' - y' = 0$ satisfying $y(0) = 0$, $y'(0) = 0$

$y = c_1 + c_2 x^2$ from #6

$y' = 2c_2 x$

$y(0) = 0 = c_1$

$$\left.\begin{aligned} y'(0) = 0 &= 2c_2 \cdot 0 \\ 0 &= 0 \end{aligned}\right\} \Rightarrow c_2 \text{ is arbitrary.}$$

One solution would be $y = 0$

another would be $y = x^2$

4.1 #9 $y = c_1 e^x \cos x + c_2 e^x \sin x$ is a solution of $y'' - 2y' + 2y = 0$ on $(-\infty, \infty)$. If possible find solutions satisfying the conditions

a) $y(0) = 1$, $y'(0) = 0$

$y(0) = 1 = c_1 \quad y'(0) = 0 = c_1 + c_2$

$\Rightarrow c_2 = -1$

$y' = c_1 e^x \cos x - c_1 e^x \sin x + c_2 e^x \sin x + c_2 e^x \cos x$

$= c_1 e^x \cos x + c_2 e^x \cos x + c_2 e^x \sin x - c_1 e^x \sin x$

$$\boxed{y = e^x \cos x - e^x \sin x}$$

b) $y(0) = 1$, $y(\pi) = -1$

$y(0) = 1 = c_1 \quad y(\pi) = -1 = c_1 e^{\pi}(-1)$

$\Rightarrow c_1 = 1$ & $c_1 = e^{-\pi}$ $\boxed{\text{impossible}}$

c) $y(0) = 1$, $y(\pi/2) = 1$

$y(0) = 1 = c_1$, $y(\pi/2) = c_2 e^{\pi/2} = 1$

$c_2 = e^{-\pi/2}$

$\Rightarrow y = e^x \cos x + e^{x - \pi/2} \sin x$

d) $y(0) = 0$, $y(\pi) = 0$

$y(0) = 0 = c_1$

$y(\pi) = -c_1 e^{\pi} = 0$

$\Rightarrow c_1 = 0$

$\Big\}$ $c_1 = 0$ and c_2 is arbitrary

$\Rightarrow y = c_2 e^x \sin x$

4.1 #11 $(x-2)y'' + 3y = x$; $y(0) = 0$, $y'(0) = 1$

Find an interval around $x=0$ for which the given has a unique solution.

as long as $x \neq 2$, then $a_2(x) = (x-2)$ will be nonzero and conditions of Uniqueness Theorem are met.

Let I be $(-\infty, 2)$

4.1 #13 Given $y = c_1 \cos\lambda x + c_2 \sin\lambda x$ is a solution to $y'' + \lambda^2 y = 0$, determine the values of the parameter λ for which the BVP $y'' + \lambda^2 y = 0$, $y(0) = 0$, $y(\pi) = 0$ has nontrivial solutions.

$y(0) = 0 \Rightarrow 0 = c_1$

$y(\pi) = 0 \Rightarrow 0 = c_1 \cos\lambda\pi + c_2 \sin\lambda\pi$

$c_1 = 0 \Rightarrow c_2 \sin\lambda\pi = 0$

$c_2 \neq 0 \Rightarrow \sin\lambda\pi = 0$

$\Rightarrow \lambda\pi = n\pi$, n a nonzero integer

$\lambda = n$, n a nonzero integer

($n = 0 = \lambda$ would give the trivial solution $y = c_2 \cdot 0 = 0$)

4.1 #22 $f_1 = e^x$, $f_2 = e^{-x}$, $f_3 = \sinh x$.

Noting that $\sinh x = \dfrac{e^x - e^{-x}}{2}$,

$$\sinh x = \tfrac{1}{2}e^x - \tfrac{1}{2}e^{-x}$$

f_3 is just a linear combination of f_1 and f_2.
So they are <u>linearly dependent</u>.

4.1 # 15 $f_1 = x$, $f_2 = x^2$, $f_3 = 4x - 3x^2$

We can see that $4 \cdot f_1 - 3 \cdot f_2 = f_3$.
Linearly dependent.

4.1 #17 $f_1 = 5$, $f_2 = \cos^2 x$, $f_3 = \sin^2 x$

We can see that $5 \cdot f_2 + 5 \cdot f_3 = f_1$
since

$$\begin{aligned} & 5 \cdot \cos^2 x + 5 \sin^2 x \\ &= 5(\cos^2 x + \sin^2 x) \\ &= 5 \\ &= f_1 \end{aligned}$$

Linearly dependent

4.1 #19 $f_1 = x$, $f_2 = x - 1$, $f_3 = x + 3$.

suspect linearly dependent: Try to find c_1, c_2, c_3
so that $c_1 x + c_2(x-1) + c_3(x+3) = 0$

$$(c_1 + c_2 + c_3)x - c_2 + 3c_3 = 0$$

$\Rightarrow$ $c_1 + c_2 + c_3 = 0$, $c_2 = 3c_3$, let $c_3 = 1 \Rightarrow c_2 = 3$

$$\begin{aligned} c_1 &= -c_2 - c_3 \\ &= -3 - 1 = -4 \end{aligned}$$

$-4(x) + 3(x-1) + 1(x+3) = 0$ ✓

4.1 #21 $f_1(x) = 1+x$, $f_2(x) = x$, $f_3(x) = x^2$

Suspect linear independence since x is not just a multiple of $x+1$ nor is x^2 a ~~co~~ linear combination of x and $x+1$.

$$W = \begin{vmatrix} x & 1+x & x^2 \\ 1 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2\begin{vmatrix} x & 1+x \\ 1 & 1 \end{vmatrix}$$

$$= 2[x-1-x]$$

$$= -2$$

$$\neq 0 \quad \therefore \text{ Linearly independent}$$

4.1 #23 $W(x^{1/2}, x^2)$ on $(0,\infty)$ To show linear independence

$$W(x^{1/2}, x^2) = \begin{vmatrix} x^{1/2} & x^2 \\ \frac{1}{2}x^{-1/2} & 2x \end{vmatrix} = 2x^{3/2} - \frac{1}{2}x^{3/2}$$

$$= \frac{3}{2}x^{3/2}$$

$$\neq 0 \text{ on } (0,\infty)$$

$\therefore$ Linearly independent.

4.1 #25 $W(\sin x, \csc x) = \begin{vmatrix} \sin x & \csc x \\ \cos x & -\csc x \cot x \end{vmatrix}$ on $(0,\pi)$

$$= -\sin x \csc x \cot x - \cos x \csc x$$

$$= -\cot x - \cot x$$

$$= -2\cot x$$

$$\neq 0 \quad \therefore \text{ Linearly independent}$$

4.1 #27 $W(e^x, e^{-x}, e^{4x}) = \begin{vmatrix} e^x & e^{-x} & e^{4x} \\ e^x & -e^{-x} & 4e^{4x} \\ e^x & e^{-x} & 16e^{4x} \end{vmatrix}$

$$= e^x \begin{vmatrix} -e^{-x} & 4e^{4x} \\ e^{-x} & 16e^{4x} \end{vmatrix} - e^x \begin{vmatrix} e^{-x} & e^{4x} \\ e^{-x} & 16e^{4x} \end{vmatrix} + e^x \begin{vmatrix} e^{-x} & e^{4x} \\ -e^{-x} & 4e^{4x} \end{vmatrix}$$

$$= e^x\left[-16e^{3x} - 4e^{3x}\right] - e^x\left[16e^{3x} - e^{3x}\right] + e^x\left[4e^{3x} + e^{3x}\right]$$

$$= e^x\left[-20e^{3x} - 15e^{3x} + 5e^{3x}\right]$$

$$= -30e^{4x}$$

$\not\equiv 0$ on $(-\infty, \infty)$ $\therefore$ Linearly Independent

4.1 #29

$W(2, e^x) = \begin{vmatrix} 2 & e^x \\ 0 & e^x \end{vmatrix} = 2e^x \not\equiv 0$ on $(-\infty, \infty)$

$\therefore$ Linearly independent.

$c_1 f_1 + c_2 f_2 = 0$ must be valid for all x on I to show linearly dependent.

4.1 #33 Verify that the given functions form a fundamental set of solutions of the diff eq on I. Form the general solution.

$y'' - y' - 12y = 0 \; ; \; e^{-3x}, \; e^{4x}, \; (-\infty, \infty)$

$y_1 = e^{-3x}, \quad y_2 = e^{4x}$

$y_1' = -3e^{-3x} \quad y_2' = 4e^{4x}$

$y_1'' = 9e^{-3x} \quad y_2'' = 16e^{4x}$

$9e^{-3x} + 3e^{-3x} - 12e^{-3x} = 0$ ✓

and $16e^{4x} - 4e^{4x} - 12e^{4x} = 0$ ✓

$$W(y_1, y_2) = \begin{vmatrix} e^{-3x} & e^{4x} \\ -3e^{-3x} & 4e^{4x} \end{vmatrix} = 4e^{x} + 3e^{x}$$

$= 7e^{x}$

$\neq 0$: linearly independent

Thus they form a fundamental set.

4.1 #37 $x^2y'' - 6xy' + 12y = 0$ $\qquad x^3, \; x^4, \qquad (0, \infty)$

$$y_1 = x^3 \qquad y_2 = x^4$$
$$y_1' = 3x^2 \qquad y_2' = 4x^3$$
$$y_1'' = 6x \qquad y_2'' = 12x^2$$

$$x^2 \cdot 6x - 6x \cdot 3x^2 + 12 \cdot x^3 \overset{?}{=} 0$$
$$6x^3 - 18x^3 + 12x^3 \overset{\checkmark}{=} 0$$

$$12x^4 - 6x \cdot 4x^3 + 12x^4 \overset{?}{=} 0$$
$$12x^4 - 24x^4 + 12x^4 \overset{\checkmark}{=} 0$$

$$W = \begin{vmatrix} x^3 & x^4 \\ 3x^2 & 4x^3 \end{vmatrix} = 4x^6 - 3x^6$$
$$= x^6$$
$$\neq 0 \text{ on } (0, \infty)$$

$\therefore$ Linearly independent

4.1 #41

$$y'' - 7y' + 10y = 24e^{x}$$

Verify $y = \underbrace{c_1 e^{2x} + c_2 e^{5x}}_{y_c} + \underset{\uparrow \atop y_p}{6e^{x}}$, $(-\infty, \infty)$ is a solution (general)

$$y_c' = 2c_1 e^{2x} + 5c_2 e^{5x}$$

$$y_c'' = 4c_1 e^{2x} + 25c_2 e^{5x}$$

$$4c_1 e^{2x} + 25c_2 e^{5x} - 14c_1 e^{2x} - 35c_2 e^{5x} + 10c_1 e^{2x} + 10c_2 e^{5x} \overset{?}{=} 0$$

$$0 \overset{\checkmark}{=} 0$$

$$W = \begin{vmatrix} e^{2x} & e^{5x} \\ 2e^{2x} & 5e^{5x} \end{vmatrix} = 5e^{7x} - 2e^{7x}$$

$$= 3e^{7x} \neq 0 \text{ on } (-\infty, \infty) \quad \therefore \text{ Linearly Ind.}$$

$$\left.\begin{aligned} y_p &= 6e^{x} \\ y_p' &= 6e^{x} \\ y_p'' &= 6e^{x} \end{aligned}\right\} \quad \begin{aligned} 6e^{x} - 42e^{x} + 60e^{x} &\overset{?}{=} 24e^{x} \\ 6e^{x} + 18e^{x} &= 24e^{x} \\ 24e^{x} &\overset{\checkmark}{=} 24e^{x} \end{aligned}$$