4.2#1 $y'' + 5y' = 0$; $y_1 = 1$

since y is missing let $u = y'$

then $y'' = u'$

and $u' + 5u = 0$

$$\mu(x) = e^{\int 5\,dx} = e^{5x}$$

$$e^{5x}u' + \cancel{5e^{5}}u = 0$$

$$d[ue^{5x}] = 0$$

$$ue^{5x} = C_1$$

$$u = C_1 e^{-5x}$$

$$y' = C_1 e^{-5x}$$

$$y = -\frac{C_1}{5} e^{-5x} + C_2$$

Let $C_1 = -5$ and $C_2 = 0$

$y = e^{-5x}$ is a second solution

4.2 #3 $y'' - 4y' + 4y = 0$; $y_1 = e^{2x}$

$$y = ue^{2x}$$

$$y' = 2ue^{2x} + u'e^{2x}$$

$$y'' = 4ue^{2x} + 2u'e^{2x} + 2u'e^{2x} + u''e^{2x}$$

$$= 4ue^{2x} + 4u'e^{2x} + u''e^{2x}$$

substituting:

$$4ue^{2x} + \cancel{4u'e^{2x}} + u''e^{2x} - 8ue^{2x} - \cancel{4u'e^{2x}} + 4ue^{2x} = 0$$

$$\left.\begin{aligned} u'' &= 0 \\ u' &= C_1 \\ u &= C_1 x + C_2 \end{aligned}\right\}$$

$C_1 = 1$, $C_2 = 0$ $\Rightarrow$

$y = xe^{2x}$ is another solution

4.2 #5 $y'' + 16y = 0$ $y_1 = \cos 4x$

Letting $y = u \cos 4x$

$y' = u' \cos 4x - 4u \sin 4x$

$y'' = u'' \cos 4x - 4u' \sin 4x - 16u \cos 4x - 4u' \sin 4x$

$= u'' \cos 4x - 8u' \sin 4x - 16u \cos 4x$

substituting:

$u'' \cos 4x - 8u' \sin 4x - 16u \cos 4x + 16u \cos 4x = 0$

$u'' \cos 4x - 8u' \sin 4x = 0$

Let $u' = w$

Then $w' \cos 4x - 8w \sin 4x = 0$

$w' - 8 \tan 4x \cdot w = 0$

$\mu(x) = e^{-8\int \tan 4x\, dx} = e^{2 \ln|\cos 4x|}$

$= \cos^2 4x$

$\cos^2 4x\, w' - 8 \tan 4x \cos^2 4x\, w = 0$

$\cos^2 4x\, w' - 8 \sin 4x \cos 4x\, w = 0$

$d[(\cos^2 4x) w] = 0$

$w \cos^2 4x = c_1$

$w = c_1 \sec^2 4x$

$u' = c_1 \sec^2 4x$

$u = \frac{c_1}{4} \tan 4x + c_2$

let $c_1 = 4$, $c_2 = 0$ $\Rightarrow u = \tan 4x$

then $y = u \cdot \cos 4x$

$= \tan 4x \cos 4x$

$y = \sin 4x$

4.2 #7 $y'' - y = 0$; $y_1 = \cosh x$

Assume $y = u\cosh x$

$$y' = u'\cosh x + u\sinh x$$

$$y'' = u''\cosh x + u'\sinh x + u'\sinh x + u\cosh x$$

$$= u''\cosh x + 2u'\sinh x + u\cosh x.$$

<u>Substituting</u>:

$$u''\cosh x + 2u'\sinh x + u\cosh x - u\cosh x = 0$$

$$u''\cosh x + 2u'\sinh x = 0$$

Let $u' = w$

$u'' = w'$

$$w'\cosh x + 2w\sinh x = 0$$

$$w' + 2w\tanh x = 0$$

$$\mu(x) = e^{\int 2\tanh x\, dx} = e^{2\ln|\cosh x|}$$

$$= e^{\ln \cosh^2 x}$$

$$= \cosh^2 x$$

$$\cosh^2 x\, w' + 2w\tanh x \cosh^2 x = 0$$

$$\cosh^2 x\, w' + 2w\sinh x\cosh x = 0$$

$$d[w\cosh^2 x] = 0$$

$$w\cosh^2 x = c_1$$

$$w = c_1 \operatorname{sech}^2 x$$

$$u' = c_1 \operatorname{sech}^2 x$$

$$u = c_1 \tanh x + c_2$$

let $c_2 = 0$, $c_1 = 1$

then $y_2 = \tanh x \cosh x$

$= \sinh x$.

4.2 #9 $\quad 9y'' - 12y' + 4y = 0 \quad ; \quad y_1 = e^{2x/3}$

$$y'' - \frac{12}{9}y' + \frac{4}{9}y = 0$$

$$y'' - \frac{4}{3}y' + \frac{4}{9}y = 0$$

$$y_2 = y_1 \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx$$

$$= e^{2x/3} \int \frac{e^{-\int -\frac{4}{3}dx}}{e^{4x/3}}\,dx$$

$$= e^{2x/3} \int \frac{e^{\frac{4x}{3}}}{e^{4x/3}}\,dx$$

$$= e^{2x/3} \int dx$$

$$= xe^{2x/3} + ce^{2x/3}$$

let $c = 0$

then $y_2 = xe^{2x/3}$

4.2 #13 $xy'' + y' = 0$; $y_1 = \ln x$ $(x>0)$

One approach: let $w = y'$

$$xw' + w = 0$$

$$w' + \frac{1}{x} w = 0 \qquad x > 0$$

$$\mu(x) = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$$

$$\Rightarrow \quad xw' + w = 0$$

$$d[xw] = 0$$

$$xw = c_1$$

$$w = \frac{c_1}{x}$$

$$y' = \frac{c_1}{x}$$

$$y = c_1 \ln x + c_2$$

Pick $c_1 = 0$, $c_2 = 1$

then another solution is $y = 1$

Another Approach:

$$y'' + \frac{1}{x} y' = 0$$

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2} dx$$

$$= \ln x \int \frac{e^{-\int \frac{1}{x} dx}}{(\ln x)^2} dx$$

$$= \ln x \int \frac{e^{-\ln x}}{(\ln x)^2} dx$$

$$= \ln x \int \frac{1}{x} (\ln x)^{-2} dx = \ln x \left[-(\ln x)^{-1} + c \right]$$

$$= -1 + c \ln x$$

Pick $c = 0$, and let $\boxed{y_2 = 1}$

4-2 #17

$$x^2y'' - xy' + 2y = 0, \quad y_1 = x\sin(\ln x)$$

$$x > 0$$

$$y'' - \frac{1}{x}y' + \frac{2}{x^2}y = 0$$

$$y_2 = y_1 \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx$$

$$= x\sin(\ln x) \int \frac{e^{\int \frac{1}{x}dx}}{x^2 \sin^2 \ln x}\,dx$$

$$= x \sin \ln x \int \frac{e^{\ln x}}{x^2 \sin^2 \ln x}\,dx$$

$$= x \sin \ln x \int \frac{x}{x^2 \sin^2 \ln x}\,dx$$

$$= x \sin \ln x \int \frac{1}{x} \csc^2 \ln x \, dx$$

$$= x \sin(\ln x)\left(-\cot(\ln x) + c\right)$$

$$= -x \sin \ln x \, \frac{\cos \ln x}{\sin \ln x} + c x \sin \ln x$$

$$= -x \cos \ln x + c x \sin \ln x$$

$$\boxed{y_2 = x \cos \ln x}$$ let $c = 0$

4.2 #23 $\quad x^2y'' - 5xy' + 9y = 0 \qquad y_1 = x^3\ln x$

$$y'' - \frac{5}{x}y' + \frac{9}{x^2}y = 0$$

$$y_2 = y_1 \int \frac{e^{-\int P_x\,dx}}{y_1^2}\,dx$$

$$= x^3\ln x \int \frac{e^{\int \frac{5}{x}dx}}{x^6(\ln x)^2}\,dx$$

$$= x^3\ln x \int \frac{e^{5\ln x}}{x^6\ln^2 x}\,dx$$

$$= x^3\ln x \int \frac{x^5}{x^6\ln^2 x}\,dx$$

$$= x^3\ln x \int \frac{1}{x}(\ln x)^{-2}\,dx$$

$$= x^3\ln x\left(-(\ln x)^{-1} + c\right)$$

$$= -x^3 + cx^3\ln x$$

let $c = 0$

$$\boxed{y_2 = x^3}$$

4.2 #25 $\quad x^2y'' - 4xy' + 6y = 0 \; ; \quad y_1 = x^2 + x^3$

$$y'' - \frac{4}{x}y' + \frac{6}{x^2}y = 0 \qquad x > 0$$

$$y_2 = y_1 \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx$$

$$= (x^2+x^3) \int \frac{e^{\int \frac{4}{x}dx}}{(x^2+x^3)^2}\,dx$$

$$= (x^2+x^3) \int \frac{e^{4\ln x}}{(x^2+x^3)^2}\,dx$$

$$= (x^2+x^3) \int \frac{x^4}{(x^2+x^3)^2}\,dx$$

$$= (x^2+x^3) \int \frac{x^4}{x^4(1+x)^2}\,dx$$

$$= (x^2+x^3) \int (1+x)^{-2}\,dx$$

$$= (x^2+x^3)\left(-(1+x)^{-1} + c\right)$$

$$= \frac{-(x^2+x^3)}{1+x} + c(x^2+x^3)$$

$$= \frac{-x^2(1+x)}{(1+x)} + c(x^2+x^3)$$

$$\boxed{y_2 = x^2} \qquad (c=0)$$

4.1 #29 $y'' - 3(\tan x)y' = 0 \qquad y_1 = 1$

$$y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2}\,dx$$

$$= \int \frac{e^{\int 3\tan x\,dx}}{1^2}\,dx$$

$$= \int e^{-3\ln(\cos x)}\,dx$$

$$= \int \cos^{-3} x\,dx$$

$$= \int \sec^3 x\,dx$$

$u = \sec x \qquad dv = \sec^2 x\,dx$

$du = \sec x \tan x \qquad v = \tan x$

$$\sec x \tan x - \int \sec x \tan^2 x\,dx$$

$$-\int \sec x\,(-1 + \sec^2 x)\,dx$$

$$= \int \sec x\,dx - \sec^3 x\,dx$$

$$= \ln|\sec x + \tan x| - \int \sec^3 x\,dx$$

$$= \sec x \tan x + \ln|\sec x + \tan x| - \int \sec^3 x\,dx$$

$$2\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x| + C$$

$$\Rightarrow \boxed{y_2 = \sec x \tan x + \ln|\sec x + \tan x|}$$

(any multiple will work) $\qquad (C = 0)$

4.2 #31 Solve the Nonhomogeneous equation by using the given solution to the homogeneous equation

$$y'' - 4y = 2 \; ; \quad y_1 = e^{-2x}$$

$$y_2 = y_1 \int \frac{e^{\int -P(x)dx}}{y_1^2}$$

$$= e^{-2x} \int \frac{e^{\int 0 dx}}{e^{-4x}} dx$$

$$= e^{-2x} \int e^{4x} \, dx$$

$$= e^{-2x} \cdot \frac{1}{4} e^{4x} + c e^{-2x}$$

let $c = 0$ and

$$y_2 = e^{2x}$$

Note that by inspection $y = -\frac{1}{2}$ is a solution to the nonhomogeneous equation.

The general solution:

$$y = y_c + y_p$$

$$= c_1 e^{-2x} + c_2 e^{2x} - \frac{1}{2}$$