4.3 # 1 $4y'' + y' = 0$

$4m^2 + m = 0$

$m(4m+1) = 0$

$m = 0, \; m = -\frac{1}{4}$

$y = c_1 + c_2 e^{-\frac{x}{4}}$

4.3 # 3 $y'' - 36y = 0$

$m^2 - 36 = 0$

$m = \pm 6$

$y = c_1 e^{-6x} + c_2 e^{6x}$

4.3 # 5 $y'' + 9y = 0$

$m^2 + 9 = 0$

$m = \pm 3i$

$y = c_1 \cos 3x + c_2 \sin 3x$

4.3 # 7 $y'' - y' - 6y = 0$

$m^2 - m - 6 = 0$

$(m-3)(m+2) = 0$

$m = 3, \; m = -2$

$y = c_1 e^{3x} + c_2 e^{-2x}$

4.3#9 $y'' + 8y' + 16y = 0$

$m^2 + 8m + 16 = 0$

$(m+4)^2 = 0$

$m = -4$ twice

$y = c_1 e^{-4x} + c_2 x e^{-4x}$

4.3#11 $y'' + 3y' - 5y = 0$

$m^2 + 3m - 5 = 0$

$m = \frac{-3 \pm \sqrt{3^2 - 4(-5)}}{2}$

$= \frac{-3 \pm \sqrt{9+20}}{2}$

$= \frac{-3 \pm \sqrt{29}}{2}$

$y = c_1 e^{\left(\frac{-3+\sqrt{29}}{2}\right)x} + c_2 e^{\left(\frac{-3-\sqrt{29}}{2}\right)x}$

4.3#17 $3y'' + 2y' + y = 0$

$3m^2 + 2m + 1 = 0$

$m = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 3}}{6}$

$= \frac{-2 \pm \sqrt{-8}}{6}$

$= \frac{-2 \pm 2i\sqrt{2}}{6}$

$= \frac{-1 \pm i\sqrt{2}}{3}$

$y = e^{-\frac{x}{3}}\left(c_1 \cos\frac{\sqrt{2}}{3}x + c_2 \sin\frac{\sqrt{2}}{3}x\right)$

Find the 3 cube roots of 1

$r_0 = 1^{1/3}(\cos 0 + i \sin 0) = 1$

$r_1 = 1^{1/3}(\cos 120° + i \sin 120°)$
$= -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$r_2 = 1^{1/3}(\cos 240° + i \sin 240°)$
$= -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

4.3# 21 $y''' - y = 0$

$m^3 - 1 = 0$

$(m-1)(m^2+m+1) = 0$

$m = 1, \quad m = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$

$y = c_1 e^{x} + e^{-\frac{x}{2}}\left(c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right)$

4.3#27 $y''' + 3y'' + 3y' + y = 0$

$m^3 + 3m^2 + 3m + 1 = 0$

$(m+1)^3 = 0$

$m = -1$ thrice

$y = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x}$

4.3#35 $y^{(5)} + 5y^{(4)} - 2y^{(3)} - 10y'' + y' + 5y = 0$

$m^5 + 5m^4 - 2m^3 - 10m^2 + m + 5 = 0$

$m^4(m+5) - 2m^2(m+5) + (m+5) = 0$

$(m+5)(m^4 - 2m^2 + 1) = 0$

$(m+5)(m^2-1)(m^2-1) = 0$

$(m+5)(m-1)(m+1)(m-1)(m+1) = 0$

$y = c_1 e^{-5x} + c_2 e^{x} + c_3 e^{-x} + c_4 x e^{x} + c_5 x e^{-x}$

4.3#37. $y'' + 16y = 0$, $y(0) = 2$, $y'(0) = -2$

$m^2 + 16 = 0$

$m = \pm 4i$

$y = c_1 \cos 4x + c_2 \sin 4x \Rightarrow y' = -4c_1 \sin 4x + 4c_2 \cos 4x$

$y(0) = 2 = c_1$

$y'(0) = -2 = 4c_2$

$\Rightarrow c_2 = -\frac{1}{2}$

$y = 2\cos 4x - \frac{1}{2}\sin 4x$

4.3#39 $y'' + 6y' + 5y = 0$, $y(0) = 0$, $y'(0) = 3$

$m^2 + 6m + 5 = 0$

$(m+5)(m+1) = 0$

$m = -1, m = -5$

$y = c_1 e^{-x} + c_2 e^{-5x} \Rightarrow y' = -c_1 e^{-x} - 5c_2 e^{-5x}$

$y(0) = 0 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_1 = -c_2$

$y'(0) = 3 \Rightarrow -c_1 - 5c_2 = 3$

$c_2 - 5c_2 = 3$

$-4c_2 = 3$

$c_2 = -\frac{3}{4} \Rightarrow c_1 = \frac{3}{4}$

$y = \frac{3}{4}e^{-x} - \frac{3}{4}e^{-5x}$

4.3 #41 $2y'' - 2y' + y = 0$, $y(0) = -1$, $y'(0) = 0$

$$2m^2 - 2m + 1 = 0$$

$$m = \frac{2 \pm \sqrt{4-8}}{2 \cdot 2}$$

$$= \frac{2 \pm 2i}{4}$$

$$= \frac{1}{2} \pm \frac{i}{2}$$

$$y = e^{\frac{x}{2}}\left(c_1 \cos\frac{x}{2} + c_2 \sin\frac{x}{2}\right)$$

$$y' = \frac{1}{2}e^{x/2}\left(c_1 \cos\frac{x}{2} + c_2 \sin\frac{x}{2}\right) + e^{x/2}\left(-\frac{c_1}{2}\sin\frac{x}{2} + \frac{c_2}{2}\cos\frac{x}{2}\right)$$

$y(0) = -1 \Rightarrow c_1 = -1$

$y'(0) = 0 \Rightarrow \frac{1}{2}c_1 + \frac{1}{2}c_2 = 0$

$$-\frac{1}{2} = -\frac{1}{2}c_2$$

$$1 = c_2$$

$$y = e^{x/2}\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)$$

4.3#45 $y'' - 3y' + 2y = 0, \quad y(0)=0,\ y'(0)=1$

$m^2 - 3m + 2 = 0$

$(m-2)(m-1) = 0$

$y = c_1 e^{2x} + c_2 e^{x}$

$y(1) = c_1 e^2 + c_2 e = 0$

$c_1 e + c_2 = 0$

$c_2 = -c_1 e$

$y' = 2c_1 e^{2x} + c_2 e^{x}$

$y'(1) = 2c_1 e^2 + c_2 e = 1$

$2c_1 e^2 - c_1 e^2 = 1$

$c_1 e^2 = 1$

$c_1 = e^{-2}$

$c_2 = -e^{-2} e$

$= -e^{-1}$

$y = e^{-2} e^{2x} - e^{-1} e^{x}$

$\boxed{y = e^{2x-2} - e^{x-1}}$

4.3#47 $y''' + 12y'' + 36y' = 0, \quad y(0)=0,\ y'(0)=1,\ y''(0)=-7$

$m^3 + 12m^2 + 36m = 0$

$m(m^2 + 12m + 36) = 0$

$m(m+6)^2$

$y = c_1 + c_2 e^{-6x} + c_3 x e^{-6x} \Rightarrow y' = -6c_2 e^{-6x} + c_3 e^{-6x} - 6c_3 x e^{-6x}$

$y(0) = c_1 + c_2 = 0, \quad \Rightarrow y'' = 36c_2 e^{-6x} - 6c_3 e^{-6x} + 36c_3 x e^{-6x} - 6c_3 e^{-6x}$

$y'(0) = -6c_2 + c_3 = 1$

$y''(0) = 36c_2 - 12c_3 = -7$

$= 36c_2 e^{-6x} - 12c_3 e^{-6x} + 36c_3 x e^{-6x}$

$-36c_2 + 6c_3 = 6$

$-6c_3 = -1$

$c_3 = \frac{1}{6} \Rightarrow -6c_2 = -c_3 + 1$

$-6c_2 = 1 - \frac{1}{6}$

$c_2 = -\frac{5}{36}$

$c_1 = -c_2 = \frac{5}{36}$

$\boxed{y = \frac{5}{36} - \frac{5}{36} e^{-6x} + \frac{1}{6} x e^{-6x}}$

#51 $y^{(4)} - 3y^{(3)} + 3y^{(2)} - y' = 0$, $y(0) = y'(0) = 0, y''(0) = y'''(0) = 1$

$m^4 - 3m^3 + 3m^2 - m = 0$

$m(m^3 - 3m^2 + 3m - 1) = 0$

1	1	−3	3	−1
		1	−2	1
	1	−2	1	0

$m(m-1)(m^2 - 2m + 1) = 0$

$m(m-1)(m-1)(m-1) = 0$

$y = C_1 + C_2 e^x + C_3 x e^x + C_4 x^2 e^x$

① $y(0) = \boxed{0 = C_1 + C_2}$

$y' = C_2 e^x + C_3 e^x + C_3 x e^x + 2C_4 x e^x + C_4 x^2 e^x$

② $y'(0) = \boxed{0 = C_2 + C_3}$

$y'' = C_2 e^x + C_3 e^x + C_3 e^x + C_3 x e^x + 2C_4 e^x + 2C_4 x e^x + 2C_4 x e^x + C_4 x^2 e^x$

③ $y''(0) = \boxed{1 = C_2 + 2C_3 + 2C_4}$

$y''' = C_2 e^x + C_3 e^x + C_3 e^x + C_3 x e^x + 2C_4 e^x + 2C_4 e^x + 2C_4 x e^x + 2C_4 x e^x + 2C_4 e^x + 2C_4 x e^x + C_4 x^2 e^x$

④ $y'''(0) = \boxed{1 = C_2 + 3C_3 + 6C_4}$

④ − ③ $\Rightarrow$ $0 = C_3 + 4C_4$

④ − ② $\Rightarrow$ $1 = 2C_3 + 6C_4$

$\Rightarrow$ $1 = -2C_4$

$-\frac{1}{2} = C_4 \Rightarrow C_3 = -4C_4 = 2 \Rightarrow C_2 = -2 \Rightarrow C_1 = 2$

$$\boxed{y = 2 - 2e^x + 2xe^x - \frac{1}{2}x^2 e^x}$$

4.3#55 $y'' + y = 0$, $y'(0) = 0$, $y'\left(\frac{\pi}{2}\right) = 2$

$m^2 + 1 = 0$

$m = \pm i$

$y = c_1 \cos x + c_2 \sin x$

$y' = -c_1 \sin x + c_2 \cos x$

$y'(0) = 0 \Rightarrow c_2 = 0$

$y'\left(\frac{\pi}{2}\right) = 2 \Rightarrow -c_1 = 2$

$y = -2 \cos x$

4.3#59 $y''' - 9y'' + 25y' - 17y = 0$; $y_1 = e^x$.

$m^3 - 9m^2 + 25m - 17 = 0$

$y = e^x$ is a solution so $m = 1$ is a root of auxiliary equation.

1	1	−9	25	−17
		1	−8	17
	1	−8	17	0

$m^2 - 8m + 17 = 0$

$m = \frac{8 \pm \sqrt{64 - 4(17)}}{2} = \frac{8 \pm 2i}{2} = 4 \pm i$

$y = c_1 e^x + e^{4x}(c_2 \cos x + c_3 \sin x)$

4.3 #65 $y^{(4)} + y = 0$

$m^4 + 1 = 0$

$(m^2 - i)(m^2 + i) = 0$

① $m = \pm\sqrt{i}$ $\quad m = \pm\sqrt{-i}$

$m^4 + 1 = 0$

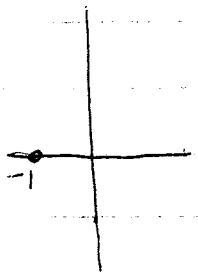
$m^2 = \dfrac{0 \pm \sqrt{0^2 - 4}}{2}$

$= \dfrac{\pm 2i}{2}$

$m^2 = \pm i$

$m = \pm\sqrt{\pm i}$

$m^4 = -1$ use complex number theory to get the 4th roots of -1



$m_0 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$
$= \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

$m_1 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$
$= -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

$m_2 = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}$
$= -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$

$m_3 = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}$
$= \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$

$$y = e^{\frac{\sqrt{2}}{2}x}\left(C_1\cos\tfrac{\sqrt{2}}{2}x + C_2\sin\tfrac{\sqrt{2}}{2}x\right) + e^{-\frac{\sqrt{2}}{2}x}\left(C_3\cos\tfrac{\sqrt{2}}{2}x + C_4\sin\tfrac{\sqrt{2}}{2}x\right)$$

Also $\quad i = \left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)^2 \quad -i = \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)^2$

$\pm\sqrt{i} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}, \quad \pm\sqrt{-i} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$

So From ① we have the 4 roots of auxiliary eq.