4.4 #21 $y''' - 6y'' = 3 - \cos x \longrightarrow y_p = A + B\cos x + C\sin x$

$m^3 - 6m^2 = 0$

$m^2(m-6) = 0$

$m=0, \; m=6$

$y_c = c_1 + c_2 x + c_3 e^{6x} \longrightarrow y_p = Ax^2 + B\cos x + C\sin x$

$y_p' = 2Ax - B\sin x + C\cos x$

$y_p'' = 2A - B\cos x - C\sin x$

$y_p''' = B\sin x - C\cos x$

$B\sin x - C\cos x - 6(2A - B\cos x - C\sin x) = 3 - \cos x$

$B\sin x - C\cos x - 12A + 6B\cos x + 6C\sin x = 3 - \cos x$

$(B+6C)\sin x + (6B - C)\cos x - 12A = 3 - \cos x$

$B + 6C = 0, \quad 6B - C = -1, \quad -12A = 3$

$B = -6C \quad 6(-6C) - C = -1 \quad A = -1/4$

$-37C = -1$

$C = \frac{1}{37} \Rightarrow B = -\frac{6}{37}$

$y = c_1 + c_2 x + c_3 e^{6x} - \frac{1}{4}x^2 - \frac{6}{37}\cos x + \frac{1}{37}\sin x$

4.4 #23 $y''' - 3y'' + 3y' - y = x - 4e^x \searrow$

$(m-1)^3 = 0$

$y_p = Ax + B + 4Ce^x$

$y_c = C_1e^x + C_2xe^x + C_3x^2e^x \longrightarrow y_p = Ax + B + 4Cx^3e^x$

(used a 4C by accident only C is needed)

$y_p' = A + 12Cx^2e^x + 4Cx^3e^x$

$y_p'' = 24Cxe^x + 12Cx^2e^x + 12Cx^2e^x + 4Cx^3e^x$

$= 24Cxe^x + 24Cx^2e^x + 4Cx^3e^x$

$y_p''' = 24Ce^x + 24Cxe^x + 48Cxe^x + 24Cx^2e^x + 12Cx^2e^x + 4Cx^3e^x$

$= 24Ce^x + 72Cxe^x + \quad Cx^2e^x + 4Cx^3e^x$

$y''' - 3y'' + 3y' - y = x - 4e^x$

$24Ce^x + \cancel{72Cxe^x} + \cancel{36Cx^2e^x} + \cancel{4Cx^3e^x} - \cancel{72Cxe^x} - \cancel{72Cx^2e^x} - \cancel{12Cx^3e^x}$

$+ 3A + \cancel{36Cx^2e^x} + \cancel{12Cx^3e^x} - Ax - B - \cancel{4Cx^3e^x} = x - 4e^x$

$24C = -4$, $\quad -A = 1$, $\quad 3A - B = 0$

$C = \frac{-1}{6}$ $\quad A = -1$ $\quad B = 3A$

$B = -3$

$y = C_1e^x + C_2xe^x + C_3x^2e^x - x - 3 - \frac{2}{3}x^3e^x$

4.4 #27 $y'' + y = 8\sin^2 x = 8\left(\frac{1-\cos 2x}{2}\right) = 4 - 4\cos 2x$

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$m^2 + 1 = 0$

$m = \pm i$

$y_c = c_1 \cos x + c_2 \sin x$

$y_p = A + B\cos 2x + C\sin 2x$

$y_p' = -2B\sin 2x + 2C\cos 2x$

$y_p'' = -4B\cos 2x - 4C\sin 2x$

$-4B\cos 2x - 4C\sin 2x + 4 + B\cos 2x + C\sin 2x = 4 - 4\cos 2x$

$A - 3B\cos 2x - 3C\sin 2x = 4 - 4\cos 2x$

$A = 4,\quad -3B = -4,\quad -3C = 0$

$B = \frac{4}{3} \qquad c = 0$

$$y = c_1\cos x + c_2 \sin x + \underbrace{4 + \frac{4}{3}\cos 2x}_{y_p}$$

4.4 # 29. $y'' + 4y = -2, \quad y\left(\frac{\pi}{8}\right) = \frac{1}{2}, \quad y'\left(\frac{\pi}{8}\right) = 2$

$m^2 + 4 = 0$

$m = \pm 2i$

$y_c = c_1 \cos 2x + c_2 \sin 2x$

$y_p = A$

$4A = -2$

$A = -\frac{1}{2}$

$y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{2}$

$y\left(\frac{\pi}{8}\right) = \frac{1}{2} = c_1 \cos\frac{\pi}{4} + c_2 \sin\frac{\pi}{4} - \frac{1}{2}$

① $1 = \frac{c_1\sqrt{2}}{2} + c_2\frac{\sqrt{2}}{2}$

$y' = -2c_1 \sin 2x + 2c_2 \cos 2x$

$y'\left(\frac{\pi}{8}\right) = 2 = -2c_1 \sin\frac{\pi}{4} + 2c_2 \cos\frac{\pi}{4}$

② $1 = -\frac{c_1\sqrt{2}}{2} + \frac{c_2\sqrt{2}}{2}$

① + ② $\Rightarrow$ $2 = \frac{2c_2\sqrt{2}}{2}$

$\frac{2}{\sqrt{2}} = c_2$

$\frac{2\sqrt{2}}{2} = c_2$

$\sqrt{2} = c_2$

① $\Rightarrow$ $2 = c_1\sqrt{2} + 2$

$c_1 = 0$

$y = \sqrt{2} \sin 2x - \frac{1}{2}$

4.4 # 31 $5y'' + y' = -6x$ $y(0)=0,\ y'(0)=-10$

$5m^2 + m = 0$

$m(5m+1) = 0$

$m=0,\quad m=-\frac{1}{5}$

$y_c = c_1 + c_2 e^{\frac{-x}{5}}$

$y_p = Ax + B$

$y_p = Ax^2 + Bx$

$y_p' = 2Ax + B$

$y_p'' = 2A$

$5y'' + y' = -6x$

$10A + 2Ax + B = -6x$

$10A + B = 0,\quad 2A = -6$

$A = -3 \Rightarrow -30 + B = 0$

$B = 30$

$y = c_1 + c_2 e^{-x/5} - 3x^2 + 30x \Rightarrow y' = -\frac{c_2}{5} e^{-x/5} - 6x + 30$

$y(0) = c_1 + c_2 = 0,\quad y'(0) = -\frac{c_2}{5} + 30 = -10$

$c_2 = -c_1$

$-c_2 = -50 - 150$

$c_2 = 200$

$y = 200e^{-x/5} - 3x^2 + 30x - 200$

4.4 #37 $y''+y = \cos x - \sin 2x$, $y\left(\frac{\pi}{2}\right)=0$, $y'\left(\frac{\pi}{2}\right)=0$

$m^2+1=0$

$m=\pm i$

$y_c = c_1\cos x + c_2\sin x$

$y_p = A\cos x + B\sin x + C\sin 2x + D\cos 2x$

use $y_p = Ax\cos x + Bx\sin x + C\sin 2x + D\cos 2x$

$$y_p' = A\cos x - Ax\sin x + B\sin x + Bx\cos x + 2C\cos 2x - 2D\sin 2x$$

$$y_p'' = -A\sin x - A\sin x - Ax\cos x + B\cos x - Bx\sin x + B\cos x - 4C\sin 2x - 4D\cos 2x$$

$y_p''+y$ becomes

$$-2A\sin x - Bx\sin x - Ax\cos x + 2B\cos x - 4C\sin 2x - 4D\cos 2x + Ax\cos x + Bx\sin x + C\sin 2x + D\cos 2x = \cos x - \sin 2x$$

$$-2A\sin x + 2B\cos x - 3C\sin 2x - 3D\cos 2x = \cos x - \sin 2x$$

$-2A=0$, $A=0$; $2B=1$, $B=\frac{1}{2}$; $-3C=-1$, $C=\frac{1}{3}$; $-3D=0$, $D=0$

$$y = c_1\cos x + c_2\sin x + \frac{1}{2}x\sin x + \frac{1}{3}\sin 2x$$

$y\left(\frac{\pi}{2}\right) = c_2 + \frac{\pi}{4} = 0$

$c_2 = -\frac{\pi}{4}$

$$y' = -c_1\sin x + c_2\cos x + \frac{1}{2}\sin x + \frac{1}{2}x\cos x + \frac{2}{3}\cos 2x$$

$y'\left(\frac{\pi}{2}\right) = -c_1 + \frac{1}{2} - \frac{2}{3} = 0$

$c_1 = \frac{1}{2} - \frac{2}{3} = \frac{-1}{6}$

$$y = \frac{-1}{6}\cos x - \frac{\pi}{4}\sin x + \frac{1}{2}x\sin x + \frac{1}{3}\sin 2x$$

4.4 #41 $y'' + y = x^2 + 1, \quad y(0) = 5, \quad y(1) = 0$

$m^2 + 1 = 0$

$m = \pm i$

$y = c_1 \cos x + c_2 \sin x$

$\Rightarrow y_p = Ax^2 + Bx + C$

$y_p' = 2Ax + B$

$y_p'' = 2A$

$2A + Ax^2 + Bx + C = x^2 + 1$

$2A + C = 1, \quad A = 1, \quad B = 0$

$C = -1$

$y_p = x^2 - 1$

$y = c_1 \cos x + c_2 \sin x + x^2 - 1$

$y(0) = 5 = c_1 - 1$

$6 = c_1$

$y(1) = c_1 \cos 1 + c_2 \sin 1 = 0$

$6 \cos 1 + c_2 \sin 1 = 0$

$c_2 \sin 1 = -6 \cos 1$

$c_2 = -\dfrac{6 \cos 1}{\sin 1}$

$= -6 \cot 1$

$y = 6 \cos x - 6(\cot 1) \sin x + x^2 - 1$