Mth 205 ZILC 5th Monbrod

Solutions to 4.7

4.7#1 $y'' + y = \sec x$

$m^2 + 1 = 0$

$m = \pm i$

$y_c = C_1 \cos x + C_2 \sin x$

Assume $y_p = u_1 y_1 + u_2 y_2$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$W_1 = \begin{vmatrix} 0 & \sin x \\ \sec x & \cos x \end{vmatrix} = -\sec x \sin x$$

$$W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \end{vmatrix} = \cos x \sec x = 1$$

$u_1' = \dfrac{W_1}{W} = -\sec x \sin x$ $\qquad u_2' = \dfrac{W_2}{W} = 1$

$u_1 = \displaystyle\int -\frac{\sin x}{\cos x}\,dx = \ln|\cos x|$ $\qquad u_2 = x$

$y_p = u_1 y_1 + u_2 y_2 = \cos x \ln|\cos x| + x \sin x$

$y = c_1 \cos x + c_2 \sin x + x \sin x + \cos x \ln|\cos x|$

47 #3 $\quad y'' + y = \sin x$

$$m^2 + 1 = 0$$

$$m = \pm i$$

$$y_c = c_1 \cos x + c_2 \sin x$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$W_1 = \begin{vmatrix} 0 & \sin x \\ \sin x & \cos x \end{vmatrix} = -\sin^2 x \qquad W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \sin x \end{vmatrix} = \cos x \sin x$$

$$u_1' = \frac{W_1}{W} = \frac{-\sin^2 x}{1} = -\sin^2 x \qquad u_2' = \frac{W_2}{W} = \frac{\cos x \sin x}{1}$$

$$u_1 = \int -\sin^2 x \, dx = -\int \frac{1 - \cos 2x}{2} dx \qquad u_2 = \frac{\sin^2 x}{2}$$

$$= -\frac{x}{2} + \frac{1}{4}\sin 2x$$

$$y_p = \left(-\frac{x}{2} + \frac{1}{4}\sin 2x\right)\cos x + \sin x \cdot \frac{\sin^2 x}{2}$$

$$= -\frac{x}{2}\cos x + \frac{1}{4}\sin 2x \cos x + \frac{1}{2}\sin^3 x$$

$$y = c_1 \cos x + c_2 \sin x - \frac{x}{2}\cos x + \frac{1}{4}\sin 2x \cos x + \frac{1}{2}\sin^3 x$$

$$= c_1 \cos x + c_2 \sin x - \frac{x}{2}\cos x + \frac{1}{2}\sin x \cos^2 x + \frac{1}{2}\sin x(1 - \cos^2 x)$$

$$= c_1 \cos x + c_2 \sin x - \frac{x}{2}\cos x + \frac{1}{2}\sin x \cos^2 x + \frac{1}{2}\sin x - \frac{1}{2}\sin x \cos^2 x$$

$$= c_1 \cos x + c_2 \sin x - \frac{x}{2}\cos x - \frac{1}{2}\sin x$$

$$= c_1 \cos x + c_3 \sin x - \frac{x}{2}\cos x$$

(combine $c_2 \sin x - \frac{1}{2}\sin x$)
$c_3 \sin x$

4.7 #5 $\quad y'' + y = \cos^2 x$

$$m^2 + 1 = 0$$

$$m = \pm i$$

$$y_c = c_1 \cos x + c_2 \sin x$$

$$y = u_1 y_1 + u_2 y_2$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1 \qquad W_1 = \begin{vmatrix} 0 & \sin x \\ \cos^2 x & \cos x \end{vmatrix} = -\sin x \cos^2 x$$

$$W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \cos^2 x \end{vmatrix} = \cos^3 x$$

$$u_1' = \frac{W_1}{W} = -\sin x \cos^2 x \quad , \quad u_2' = \frac{W_2}{W} = \cos^3 x$$

$$u_1 = \int \cos^2 x \sin x \, dx \qquad u_2 = \int \cos^3 x \, dx$$

$$u_1 = \frac{\cos^3 x}{3} \qquad = \int \cos^2 x \cos x \, dx$$

$$= \int (1 - \sin^2 x) \cos x \, dx$$

$$= \int \cos x - \sin^2 x \cos x \, dx$$

$$= \sin x - \frac{\sin^3 x}{3}$$

$$y_p = \frac{\cos^3 x}{3} \cdot \cos x + \left(\sin x - \frac{\sin^3 x}{3}\right) \sin x$$

$$= \frac{\cos^4 x}{3} + \sin^2 x - \frac{\sin^4 x}{3}$$

$$y = c_1 \cos x + c_2 \sin x + \frac{\cos^4 x}{3} - \frac{\sin^4 x}{3} + \sin^2 x$$

4.7 #5 continued

$$y = c_1\cos x + c_2\sin x + \frac{1}{3}(\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x) + \sin^2 x$$
$$= c_1\cos x + c_2\sin x + \frac{1}{3}\cos^2 x - \frac{1}{3}\sin^2 x + \sin^2 x$$
$$= c_1\cos x + c_2\sin x + \frac{1}{3}(1-\sin^2 x) + \frac{2}{3}\sin^2 x$$
$$= c_1\cos x + c_2\sin x + \frac{1}{3} + \frac{1}{3}\sin^2 x$$

4.7 #11 $\quad y'' + 3y' + 2y = \dfrac{1}{1+e^x}$

$m^2 + 3m + 2 = 0$

$(m+2)(m+1)$

$m=-2,\ m=-1$

$y_c = c_1 e^{-2x} + c_2 e^{-x}$

$y = u_1 y_1 + u_2 y_2$

$$W = \begin{vmatrix} e^{-2x} & e^{-x} \\ -2e^{-2x} & -e^{-x} \end{vmatrix} \qquad W_1 = \begin{vmatrix} 0 & e^{-x} \\ \frac{1}{1+e^x} & -e^{-x} \end{vmatrix} = \frac{-e^{-x}}{1+e^x}$$

$$= -e^{-3x} + 2e^{-3x} \qquad W_2 = \begin{vmatrix} e^{-2x} & 0 \\ -2e^{-2x} & \frac{1}{1+e^x} \end{vmatrix} = \frac{e^{-2x}}{1+e^x}$$
$$= e^{-3x}$$

$$u_1' = \frac{W_1}{W} = \frac{-e^{-x}}{e^{-3x}(1+e^x)} = \frac{-e^{2x}}{1+e^x}$$

$$u_1 = \int \frac{-e^{2x}}{1+e^x}\,dx \overset{\text{Divide}}{\Longrightarrow} -\int \left[e^x - \frac{e^x}{1+e^x}\right]dx = -e^x + \ln(1+e^x)$$

$$u_2' = \frac{W_2}{W} = \frac{e^{-2x}}{e^{-3x}(1+e^x)} = \frac{e^x}{1+e^x}$$

$$u_2 = \ln(1+e^x)$$

4-7 #11 continued

$$y = c_1 e^{-2x} + c_2 e^{-x} + e^{-2x}(-e^{x} + \ln(1+e^{x})) + e^{-x}(\ln(1+e^{x}))$$
$$= c_1 e^{-2x} + c_2 e^{-x} - e^{-x} + e^{-2x}\ln(1+e^{x}) + e^{-x}\ln(1+e^{x})$$
$$= c_1 e^{-2x} + c_3 e^{-x} + e^{-x}(e^{-x}+1)\ln(1+e^{x})$$

4-7 #17

$$y'' + 2y' + y = e^{-x}\ln x$$
$$m^2 + 2m + 1 = 0$$
$$(m+1)^2$$
$$m = -1$$
$$y_c = c_1 e^{-x} + c_2 x e^{-x}$$
$$y_p = u_1 y_1 + u_2 y_2$$
$$W = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x} - xe^{-x} \end{vmatrix} = e^{-2x} - xe^{-2x} + xe^{-2x}$$
$$= e^{-2x}$$
$$W_1 = \begin{vmatrix} 0 & xe^{-x} \\ e^{-x}\ln x & e^{-x} - xe^{-x} \end{vmatrix} = -xe^{-2x}\ln x$$
$$W_2 = \begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & e^{-x}\ln x \end{vmatrix} = e^{-2x}\ln x$$
$$u_1' = \frac{W_1}{W} = \frac{-xe^{-2x}\ln x}{e^{-2x}} = -x\ln x$$
$$u_1 = \int -x\ln x\, dx = -\frac{x^2}{2}\ln x + \frac{x^2}{4}$$

4.7 #17 Continued

$$u_2' = \frac{W_2}{W} = \frac{e^{-2x}\ln x}{e^{-2x}} = \ln x$$

$$u_2 = \int \ln x \, dx$$

$$= x\ln x - x$$

$$y_p = e^{-x}\left(-\frac{x^2}{2}\ln x + \frac{x^2}{4}\right) + xe^{-x}(x\ln x - x)$$

$$= x^2 e^{-x}\left(\frac{1}{4} - \frac{1}{2}\ln x\right) + x^2 e^{-x}(\ln x - 1)$$

$$= x^2 e^{-x}\left(\frac{1}{4} - \frac{1}{2}\ln x + \ln x - 1\right)$$

$$= x^2 e^{-x}\left(\frac{1}{2}\ln x - \frac{3}{4}\right)$$

$$y = c_1 e^{-x} + c_2 x e^{-x} + x^2 e^{-x}\left(\frac{1}{2}\ln x - \frac{3}{4}\right)$$

4.7 # 21. $y''' + y' = \tan x$

$m^3 + m = 0$

$m(m^2+1) = 0$

$m = 0,\ m = \pm i$

$y_c = c_1 + c_2 \cos x + c_3 \sin x$

$y_p = u_1 y_1 + u_2 y_2 + u_3 y_3$

$$W = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix} = \begin{vmatrix} -\sin x & \cos x \\ -\cos x & -\sin x \end{vmatrix} = \sin^2 x + \cos^2 x$$
$$= 1$$

$$W_1 = \begin{vmatrix} 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ \tan x & -\cos x & -\sin x \end{vmatrix} = \tan x \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \tan x(\cos^2 x + \sin$$
$$= \tan x$$

$$W_2 = \begin{vmatrix} 1 & 0 & \sin x \\ 0 & 0 & \cos x \\ 0 & \tan x & -\sin x \end{vmatrix} = \begin{vmatrix} 0 & \cos x \\ \tan x & -\sin x \end{vmatrix} = -\sin x$$

$$W_3 = \begin{vmatrix} 1 & \cos x & 0 \\ 0 & -\sin x & 0 \\ 0 & -\cos x & \tan x \end{vmatrix} = \begin{vmatrix} -\sin x & 0 \\ -\cos x & \tan x \end{vmatrix} = -\sin x \tan x$$

$$u_1' = \frac{W_1}{W} = \frac{\tan x}{1}$$

$$u_1 = \int \tan x\, dx = -\ln|\cos x| = -\ln(\cos x) \qquad 0 \le x < 1$$

$$u_2' = \frac{W_2}{W} = -\sin x$$

$$u_2 = -\int \sin x\, dx = \cos x$$

47 #21 continued

$$u_3' = \frac{W_3}{W} = -\sin x \tan x$$

$$u_3 = -\int \frac{\sin x \sin x}{\cos x}\,dx$$

$$= -\int \frac{\sin^2 x}{\cos x}\,dx$$

$$= -\int \frac{1-\cos^2 x}{\cos x}\,dx$$

$$= \int \left(\frac{-1}{\cos x} + \cos x\right) dx$$

$$= \int (-\sec x + \cos x)\,dx$$

$$= -\ln|\sec x + \tan x| + \sin x$$

$$y_p = 1(-\ln(\cos x)) + \cos x(\cos x) + \sin x(\sin x - \ln(\sec x + \tan x))$$

$$= -\ln\cos x + \cos^2 x + \sin^2 x - \sin x \ln(\sec x + \tan x)$$

$$= -\ln\cos x + 1 - \sin x \ln(\sec x + \tan x)$$

$$y = c_1 + c_2\cos x + c_3\sin x - \ln\cos x + 1 - \sin x \ln(\sec x + \tan x)$$

$$= c_4 + c_2\cos x + c_3\sin x - \ln\cos x - \sin x \ln(\sec x + \tan x)$$

$$0 \le x < \pi/2$$

4.7 #25 $\quad 4y'' - y = xe^{x/2}$ $\qquad y(0) = 1,\ y'(0) = 0$

$y'' - \frac{1}{4}y = \frac{1}{4}xe^{x/2}$

$m^2 - \frac{1}{4} = 0$

$m = \pm\frac{1}{2}$

$y_c = c_1e^{x/2} + c_2e^{-x/2}$

$$W = \begin{vmatrix} e^{x/2} & e^{-x/2} \\ \frac{1}{2}e^{x/2} & -\frac{1}{2}e^{-x/2} \end{vmatrix} = -\frac{1}{2} - \frac{1}{2} = -1$$

$y_p = u_1y_1 + u_2y_2$

$$W_1 = \begin{vmatrix} 0 & e^{-x/2} \\ \frac{1}{4}xe^{x/2} & -\frac{1}{2}e^{-x/2} \end{vmatrix}$$

$= -\frac{1}{4}x$

$$W_2 = \begin{vmatrix} e^{x/2} & 0 \\ \frac{1}{2}e^{x/2} & \frac{1}{4}xe^{x/2} \end{vmatrix} = \frac{1}{4}xe^x$$

$u_1' = \frac{W_1}{W} = \frac{-\frac{1}{4}x}{-1} = \frac{1}{4}x$

$u_1 = \frac{x^2}{8}$

$u_2' = \frac{W_2}{W} = \frac{\frac{1}{4}xe^x}{-1} = -\frac{1}{4}xe^x$

$u_2 = -\frac{1}{4}xe^x + \frac{1}{4}e^x$

$y_p = \frac{x^2}{8}e^{x/2} + \left(\frac{1}{4}e^x - \frac{1}{4}xe^x\right)e^{-x/2}$

$= \frac{x^2}{8}e^{x/2} + \frac{1}{4}e^{x/2} - \frac{1}{4}xe^{x/2}$

4.7#25 continued

$$y = y_a + y_p$$

$$= c_1 e^{x/2} + c_2 e^{-x/2} + \frac{x^2}{8} e^{x/2} + \frac{1}{4} e^{x/2} - \frac{1}{4} x e^{x/2}$$

$$= c_3 e^{x/2} + c_2 e^{-x/2} + \frac{x^2}{8} e^{x/2} - \frac{x}{4} e^{x/2}$$

$$y(0) = 1 \Rightarrow 1 = c_3 + c_2 \quad (1)$$

$$y' = \frac{c_3}{2} e^{x/2} - \frac{c_2}{2} e^{-x/2} + \frac{x}{4} e^{x/2} + \frac{x^2}{16} e^{x/2} - \frac{1}{4} e^{x/2} - \frac{x}{8} e^{x/2}$$

$$y'(0) = 0 = \frac{c_3}{2} - \frac{c_2}{2} - \frac{1}{4}$$

$$\frac{1}{4} = \frac{c_3}{2} - \frac{c_2}{2}$$

$$\frac{1}{2} = c_3 - c_2 \quad (2)$$

$$(1) + (2) \Rightarrow \frac{3}{2} = 2c_3$$

$$\frac{3}{4} = c_3 \qquad \Rightarrow \qquad c_2 = 1 - c_3 = 1 - \frac{3}{4} = \frac{1}{4}$$

$$y = \frac{3}{4} e^{x/2} + \frac{1}{4} e^{-x/2} + \frac{x^2}{8} e^{x/2} - \frac{x}{4} e^{x/2}$$