MTH 205

Solutions 5.1–5.3 Z&C 5th Monbrod

5.1 #1 $\frac{4}{32}x'' + 3x = 0$

$\frac{1}{8}x'' + 3x = 0$ $x(0) = -3,\ x'(0) = -2$

A mass ~~on~~ 1/8 a spring is lifted 3 units above equilibrium and when $t=0$, let go with a velocity of 2 units/unit time in upward direction.

See Pg 13 in text for Units Explanation

$m = \frac{4}{32} = \frac{W}{g}$

$W = 4\text{lbs}$ since $g = 32\ \text{ft}/s^2$

$m = \frac{1}{8}$ slug

Spring constant k is 3 Lb/ft.

5.1 # 3 Write the solutions in form $x = A\sin(\omega t+\phi)$

$x'' + 25x = 0$, $x(0) = -2$, $x'(0) = 10$

$x = A\sin(\omega t + \phi)$

$\omega^2 = 25$

$\omega = 5$

$x = c_1 \cos 5t + c_2 \sin 5t$

$\boxed{x(0) = -2 = c_1}$

$\left.\begin{array}{l} \sin\phi = c_1/A < 0 \\ \cos\phi = c_2/A > 0 \end{array}\right\} \phi \text{ in QIV}$

$x' = -5c_1 \sin 5t + 5c_2 \cos 5t$

$x'(0) = 10 = 5c_2$

$2 = c_2$

$x = -2\cos 5t + 2\sin 5t$

$\Rightarrow$ If $A = \sqrt{(-2)^2 + 2^2} = \sqrt{8}$, then

$x = \sqrt{8}\sin(5t + \phi)$

where $\phi = \tan^{-1}\left(\frac{-2}{2}\right) = \tan^{-1}(-1) = -\frac{\pi}{4}$

$x = \sqrt{8}\sin(5t - \frac{\pi}{4})$

5.1 # 5 $\quad x'' + 2x = 0$

$x(0) = -1, \quad x'(0) = -2\sqrt{2}$

$\omega^2 = 2$

$\omega = \sqrt{2}$

$x = c_1 \cos\sqrt{2}\,t + c_2 \sin\sqrt{2}\,t$

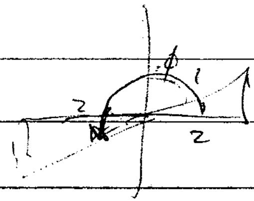
$$\left.\begin{array}{l} x(0) = -1 \Rightarrow \quad -1 = c_1 \qquad \Rightarrow \sin\phi < 0 \\ x' = -c_1\sqrt{2}\sin\sqrt{2}\,t + c_2\sqrt{2}\cos\sqrt{2}\,t \\ x'(0) = -2\sqrt{2} = c_2\sqrt{2} \\ \qquad -2 = c_2 \qquad \Rightarrow \cos\phi < 0 \end{array}\right\} \phi \text{ in Quad III}$$

$A = \sqrt{(-1)^2 + (-2)^2} = \sqrt{5}$

$\tan\phi = \frac{c_1}{c_2} = \frac{-1}{-2} = \frac{1}{2}$

so $\phi = \tan^{-1}\left(\frac{1}{2}\right) + \pi$

$= 3.61$



$x = A \sin(\omega t + \phi)$

$= \sqrt{5}\sin(\sqrt{2}\,t + 3.61)$

5.1 #9 $T = \frac{\pi}{4}$ seconds (free undamped mass on spring)

$k = 16\,Lb/ft$

Find Weight

$$T = \frac{2\pi}{\omega} = \frac{\pi}{4}$$

$$\Rightarrow \omega = 8$$

$$\omega^2 = \frac{k}{m} =$$

$$64 = \frac{16}{m}$$

$$m = \frac{16}{64}$$

$$m = \frac{1}{4}$$

$$W = mg$$

$$= \frac{1}{4} \cdot 32$$

$$= 8\,Lb$$

5.1 #9 $T = \frac{\pi}{4}$ seconds (free undamped mass on spring)

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$$\omega^2 = \frac{k}{m} =$$

$$64 = \frac{16}{m}$$

$$m = \frac{16}{64}$$

$$m = \frac{1}{4}$$

$$W = mg$$

$$= \frac{1}{4} \cdot 32$$

$$= 8 \text{ lb}$$

5.1 #11

$W = 4$ Lb

$k = 16$ Lb/ft

Find T, Period

$$T = \frac{2\pi}{\omega}$$

$$mx'' + kx = 0$$

$$mg = ks$$

$$4 = 16 \cdot s$$

$\frac{1}{4} = s$ at equilibrium

$x(0) = 0$

$$4 = mg$$

$$m = \frac{4}{32} = \frac{1}{8}$$

$$\omega^2 = \frac{k}{m}$$

$$\omega^2 = \frac{16}{1/8}$$

$$\omega = \sqrt{8 \cdot 16}$$

$$= 4 \cdot 2\sqrt{2}$$

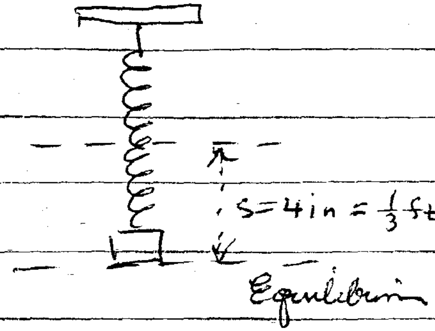
$$= 8\sqrt{2}$$

$$T = \frac{2\pi}{8\sqrt{2}}$$

$$= \frac{\pi}{4\sqrt{2}}$$

$$= \frac{\sqrt{2}\pi}{8} \text{ seconds}$$

5.1 #13 $\quad W = 24\,Lb \qquad s = 4\,in$



$$\left.\begin{aligned} x(0) &= -.25 \\ x'(0) &= 0 \end{aligned}\right\}$$ released from rest 3in (.25 ft) at $t=0$.

① $\cancel{mx'' + kx = 0}$

$W = mg$

$24 = m \cdot 32$

$\frac{24}{32} = m$

$\frac{3}{4} = m$

$\downarrow$ at equilibrium

$mg = ks$

$24 = \frac{k}{3}$

$72 = k$

$.75x'' + 72x = 0$

$x'' + \frac{4}{3} \cdot 72x = 0$

$\cancel{x'' + 96x = 0}$

$\omega^2 = 96 \Rightarrow \omega = \sqrt{96} = 4\sqrt{6}$

$x = c_1 \cos 4\sqrt{6}\,t + c_2 \sin 4\sqrt{6}\,t$

$x(0) = -\frac{1}{4} = c_1$

$x' = -4\sqrt{6}\,c_1 \sin 4\sqrt{6}\,t + 4\sqrt{6}\,c_2 \cos 4\sqrt{6}\,t$

$x'(0) = 0 \Rightarrow 4\sqrt{6}\,c_2 = 0$

$c_2 = 0$

$x = -\frac{1}{4} \cos 4\sqrt{6}\,t$

5.1 # 19 $W = 8\,Lb$ simple harmonic motion

$k = 1\,Lb/ft$

$X(0) = .5\,f$

$x'(0) = \frac{3}{2}\,f/sec$

Find Equation of motion and leave in form

$x = A\sin(\omega t + \phi)$

$W = 8 = mg$

$\frac{8}{32} = m$

$\frac{1}{4} = m$

$$\left.\begin{aligned} mx'' + kx &= 0 \\ \tfrac{1}{4}x'' + x &= 0 \\ x'' + 4x &= 0 \\ \omega^2 &= 4 \\ \omega &= 2 \end{aligned}\right\} \quad x = c_1\cos 2t + c_2\sin 2t$$

$$x' = -2c_1 \sin 2t + 2c_2 \cos 2t$$

$$x'(0) = \frac{3}{2} = 2c_2$$

$$\frac{3}{4} = c_2$$

$\cos\phi > 0$ $\Big\}$ ϕ in Quad I.

$$x = \frac{1}{2}\cos 2t + \frac{3}{4}\sin 2t$$

$$x = A\sin(\omega t + \phi)$$

$$A = \sqrt{c_1^2 + c_2^2}$$

$$= \sqrt{\frac{1}{4} + \frac{9}{16}}$$

$$= \sqrt{\frac{13}{16}} = \frac{\sqrt{13}}{4}$$

$$\phi = \tan^{-1}\left(\frac{c_1}{c_2}\right) = \frac{1/2}{3/4} = \frac{2}{3}$$

$$\phi = .5880$$

$$x = \frac{\sqrt{13}}{4}\sin(2t + .588)$$

5.1 #21 $W = 64\,lb$ 64 lb weight stretches .32 ft

$mg = ks$ $x(0) = -\frac{8}{12}\,ft = -\frac{2}{3}\,ft$

$64 = k(.32)$ $x'(0) = 5\,ft/sec$

$k = \frac{64}{.32}$

$= 200$

$W = mg$

$64 = 32m$

$2 = m$

$2x'' + 200x = 0$

$x'' + 100x = 0$ $\omega^2 = 100$, $\omega = 10$

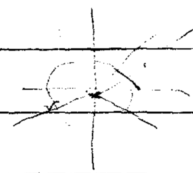
$x = c_1 \cos 10t + c_2 \sin 10t$

$x(0) = -\frac{2}{3} = c_1$

$x' = -10c_1 \sin 10t + 10c_2 \cos 10t$

$x'(0) = 5 = 10c_2$

$\frac{1}{2} = c_2$



a) $x = -\frac{2}{3}\cos 10t + \frac{1}{2}\sin 10t$

$= A\sin(10t + \phi)$ $\left.\begin{array}{ll} \sin\phi = c_1/A & <0 \\ \cos\phi = c_2/A & >0 \end{array}\right\}$ ϕ in Quad $\underline{IV}$

$A = \sqrt{c_1^2 + c_2^2} = \sqrt{\frac{1}{4} + \frac{4}{9}} = \sqrt{\frac{9+16}{36}} = \sqrt{\frac{25}{36}} = \frac{5}{6}$

$\phi = \tan^{-1}\frac{c_1}{c_2} = \tan^{-1}\frac{-2/3}{1/2} = \tan^{-1}\left(-\frac{4}{3}\right) = -.927$

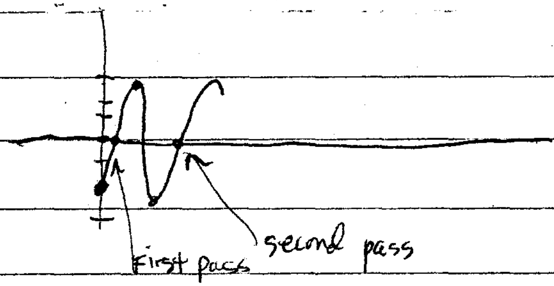
$A = \frac{5}{6}\sin(10t - .927)$ Alternate form

b) $A = \frac{5}{6}$ ft Amplitude

$T = \frac{2\pi}{\omega} = \frac{2\pi}{10} = \frac{\pi}{5}$ seconds/cycle

c) $\frac{3\pi s}{\frac{\pi}{5} s/\text{cycle}} = 15$ cycles

d) At what time does weight pass through the equilibrium position heading downward for the second time?



$0 = \frac{5}{6} \sin(10t - .927)$

$0 = \sin(10t - .927)$

$10t - .927 = 0, \pi, 2\pi, \ldots$

$= n\pi, \quad n = 0, 1, 2, \ldots$

$10t = .927 + n\pi$

$t = \frac{.927 + n\pi}{10}$

$n = 0 \quad t = \frac{.927}{10} =$

$n = 1 \quad t = \frac{.927 + \pi}{10}$

Second pass (heading Down) $n = 2 \quad t = \frac{.927 + 2\pi}{10}$

$t = .721$ seconds

e) extreme displacement on either side of equilibrium occurs when $x(t)$ has max or min.

$x'(t) = \frac{5}{6} \cdot 10 \cos(10t - .927)$

$0 = \frac{50}{6} \cos(10t - .927)$

$10t - .927 = \frac{n\pi}{2}$, $n =$ odd integer.

$10t = \frac{n\pi}{2} + .927$

$t = \frac{n\pi}{20} + .0927$, $n = 1, 3, 5, \ldots$

($n = 1$ gives First positive value of t)

$t = .2497$

f) $x(3) = \frac{5}{6} \sin(10 \cdot 3 - .927)$

$x = -.597\text{ ft}$ .597 ft above equilibrium

g) $x'(3) = \frac{50}{6} \cos(30 - .927)$

$= -5.8145\text{ ft/sec}$

h) $x''(t) = -\frac{500}{6} \sin(10t - .927)$

$= -\frac{250}{3} \sin(10t - .927)$

$x''(3) = 59.7\text{ ft/s}^2$

i) From graph, this would occur where graph crosses t axis.

$t = \frac{.927 + n\pi}{10}$

For ex. $x'\left(\frac{.927}{10}\right) = \frac{50}{6} \cos\left(10\left(\frac{.927}{10}\right) - .927\right) = \frac{50}{6}$

$x'\left(\frac{.927 + \pi}{10}\right) = \frac{50}{6} \cos\left(10\left(\frac{.927 + \pi}{10}\right) - .927\right) = -\frac{50}{6}$

$x' = \pm \frac{25}{3}\text{ ft/sec}$ Mead

j) $\sin = \frac{5}{12}\text{ ft} = \frac{5}{6}\sin(10t - .927)$

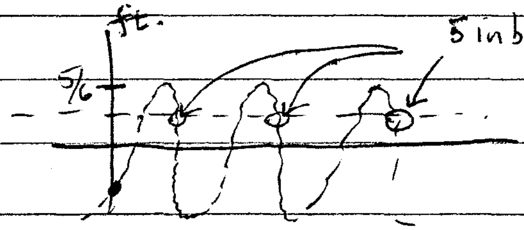
$$\frac{1}{2} = \sin(10t - .927)$$

$$10t - .927 = \frac{\pi}{6} + 2k\pi, \quad \frac{5\pi}{6} + 2k\pi \qquad k = 0, 1, 3 \ldots$$

$$t = \frac{\frac{\pi}{6} + 2k\pi + .927}{10} \qquad t = \frac{\frac{5\pi}{6} + 2k\pi + .927}{10}$$

$$= \frac{\pi + 12k\pi + 5.562}{60} \quad k = 0, 1, 2, 3 \ldots \qquad = \frac{5\pi + 12k\pi + 5.562}{60} \quad k = 0, 1, 2 \ldots$$

k)



5 in below is $x > 0$ and velocity upward means $x' < 0$ for mass.

$$t = \frac{5\pi + 12k\pi + 5.562}{60}, \quad k = 0, 1, 2, \ldots$$

1#28 $\quad x = A\sin(\omega t + \phi)$

$$x'(t) = -\omega A\cos(\omega t + \phi)$$

$$0 = -\omega A\cos(\omega t + \phi)$$

$$\cos(\omega t + \phi) = 0$$

$$\omega t + \phi = \frac{n\pi}{2}, \quad n \text{ odd integer}$$

$$t = \left(\frac{n\pi}{2} - \phi\right)\frac{1}{\omega}$$

maximum of $x'(t)$ can be found with $x''(t) = 0$.

$$x''(t) = \omega^2 A\sin(\omega t + \phi)$$

$$0 = \sin(\omega t + \phi)$$

$$\omega t + \phi = n\pi, \quad n = 0, 1, 2, 3, \ldots$$

$$t = \frac{n\pi - \phi}{\omega}$$

notice $x = 0$ for these same values of t.

Equilibrium