5.2 #1 Give a physical interpretation of the given initial value problem

$$\frac{1}{16}x'' + 2x' + x = 0$$

$x(0) = 0$, $x'(0) = -1.5$

Spring constant

$k = 1$

$\beta = 1$

Retarding force twice instant. vel.

$m = \frac{1}{16}$ slug $\quad mg = W$

$\frac{1}{16} \cdot 32 = W$

$2 = W$

2 lb weight

Mass released from equilibrium, with an upward velocity of 1.5 ft/sec

5.2 #7 $mg = 4 \quad k = 2$

$$mx'' + \beta x' + kx = 0$$

$\beta = 1$ ← retarding force equal to $x'$

$X(0) = -1$

$X'(0) = 8\ \text{ft/sec}$

Find t when mass passes through equilibrium pos. and Find the time at which the weight attains its extreme displacement from equilibrium. What is the position at this instant?

$$m = \frac{4}{32} = \frac{1}{8}$$

$$\frac{1}{8}x'' + x' + 2x = 0$$

$$x'' + 8x' + 16x = 0$$

Aux Eq: $(m+4)^2 = 0$

$$m = -4$$

$X(t) = c_1 e^{-4t} + c_2 t e^{-4t}$ (critically damped)

$$X(0) = -1 = c_1$$

$$x'(t) = 4e^{-4t} - 4c_2 t e^{-4t} + c_2 e^{-4t}$$

$$X'(0) = 8 = 4 + c_2 \Rightarrow c_2 = 4$$

$$X(t) = -e^{-4t} + 4te^{-4t}$$

$$X'(t) = 4e^{-4t} - 16te^{-4t} + 4e^{-4t}$$

$$0 = 8e^{-4t}(1 - 2t)$$

$$t = 1/2$$

$X(1/2) = -e^{-2} + 2e^{-2} = e^{-2}$ extreme disp.

$$X(t) = 0 \Rightarrow 4te^{-4t} = e^{-4t}$$

$$t = 1/4$$

#9 $m = 1\,kg$ $k = 16\,N/m$

$\beta = 10$

$mx'' + \beta x' + kx = 0$

~~Find eq. of Motion:~~

a) $x(0) = 1\,m$ b) $x(0) = 1\,m$

$x'(0) = 0$ $x'(0) = -12\,m/s$

$$x'' + 10x' + 16x = 0$$

$$(m+8)(m+2) = 0$$

$$m = -8, \quad m = -2$$

overdamped.

$$x(t) = c_1 e^{-2t} + c_2 e^{-8t}$$

$$x(0) = 1 = c_1 + c_2 \Rightarrow c_1 = 1 - c_2$$

$$x'(t) = -2c_1 e^{-2t} - 8c_2 e^{-8t}$$

$$x'(0) = 0 = 2c_1 + 8c_2 \Rightarrow 2c_1 = -8c_2$$

$$c_1 = -4c_2$$

$$1 - c_2 = -4c_2$$

$$1 = -3c_2$$

$$-1/3 = c_2 \Rightarrow c_1 = 1 + \tfrac{1}{3} = \tfrac{4}{3}$$

a) $x(t) = \frac{4}{3}e^{-2t} - \frac{1}{3}e^{-8t}$

b) $x(0) = 1 = c_1 + c_2 \Rightarrow c_1 = 1 - c_2$

$$x'(0) = -12 = -2c_1 \;\; 8c_2$$

$$12 = 2(1 - c_2) + 8c_2$$

$$6 = 1 - c_2 + \cancel{4}c_2$$

$$5 = 3c_2$$

$$\frac{5}{3} = c_2 \Rightarrow c_1 = 1 - \frac{5}{3} = -\frac{2}{3}$$

$$x = -\frac{2}{3}e^{-2t} + \frac{5}{3}e^{-8t}$$

Mead

5.2 #11 $\qquad k = 2$

$$F = kx$$
$$2 = k \cdot 1$$
$$2 = k$$

$$mg = 3.2\,lb \Rightarrow m = \frac{3.2}{32} = .1 \text{ slug}$$

$$\beta = 0.4$$

$$mx'' + \beta x' + kx = 0$$
$$.1x'' + 0.4x' + 2x = 0$$
$$x'' + 4x' + 20x = 0$$

Aux eq. $\quad m = \dfrac{-4 \pm \sqrt{16-80}}{2} = \dfrac{-4 \pm 8i}{2}$

$$= -2 \pm 4i$$

$$x(t) = e^{-2t}(c_1 \cos 4t + c_2 \sin 4t)$$

a) $\quad x(0) = -1, \quad x'(0) = 0$

$x(0) = -1 = c_1 \qquad x'(t) = -2e^{-2t}(c_1 \cos 4t + c_2 \sin 4t) + e^{-2t}(4c_2 \cos 4t - 4c_1 \sin 4t)$

$$x'(0) = 0 \Rightarrow 0 = -2c_1 + 4c_2$$
$$0 = 2 + 4c_2$$
$$c_2 = -\tfrac{1}{2}$$

$$x(t) = -e^{-2t}\left(\cos 4t + \frac{1}{2}\sin 4t\right)$$

b) Put into form $\quad x(t) = Ae^{-\lambda t}\sin(\sqrt{\omega^2 - \lambda^2}\,t + \phi)$

FROM

$$x'' + 2\lambda x' + \omega^2 x = 0$$
$$x'' + 4x' + 20x = 0$$

$2\lambda = 4, \quad \omega^2 = 20$

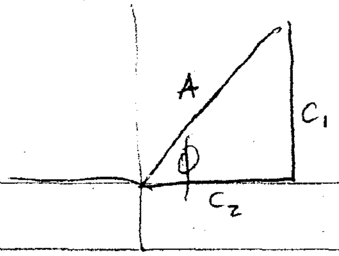
$\lambda = 2$

$\lambda^2 = 4$

Mead

52 #11 CONT.

$$A = \sqrt{c_1^2 + c_2^2}$$




$$-\cos 4t - \frac{1}{2}\sin 4t = A \cdot \sin(\sqrt{\omega^2 - \lambda^2}\, t + \phi)$$

$$= A \sin(4t + \phi)$$

$$\frac{c_1}{A} = \sin\phi = -1, \quad \frac{c_2}{A} = \cos\phi = -\tfrac{1}{2} \Rightarrow \phi \text{ in Quad III}$$

$$A = \sqrt{1 + \tfrac{1}{4}} = \sqrt{\tfrac{5}{4}} = \frac{\sqrt{5}}{2}$$

$$\frac{\sin\phi}{\cos\phi} = \tan\phi = 2$$

$$\phi = \tan^{-1}(2) = 1.107 + \pi = 4.25$$

↳ $\theta$ in Quad III

$$\therefore \quad x(t) = \frac{\sqrt{5}}{2} e^{-2t} \sin(4t + 4.25)$$

c) (when $x'(t) < 0$) find $t$ for the first pass of weight through equilibrium (heading upward.)

$$4t + 4.25 = n\pi, \quad n = 2, 3, \ldots$$

$$t = \frac{n\pi - 4.25}{4}$$

$n = 2$ gives $= \dfrac{2\pi - 4.25}{4}$

$t = .51$ seconds but $x'(t) = -\sqrt{5}e^{-2t}\sin(4t+4.25) + 2\sqrt{5}\, e^{-2t}\cos(4t+4.25)$

and $x'(.51) > 0$ (heading downward)

$n = 3$ gives $t = \dfrac{3\pi - 4.25}{4} = \boxed{1.29}$ and

$x'(1.29) < 0$ (heading upward)

5.2#13 10Lb weight stretches spring 2 ft.

$$m x'' + \beta x' + kx = 0$$

| SPRING | damping |
|---|---|
| $10 = k \cdot 2$ | $\beta x'$ |
| $5 = k$ | $(\beta > 0)$ |
| $10 = m \cdot g$ | |
| $m = \frac{10}{32} = \frac{5}{16}$ | |

$$\frac{5}{16} x'' + \beta x' + 5x = 0$$

$$x'' + \frac{16\beta}{5} x' + 16x = 0$$

$$m = \frac{-\frac{16}{5}\beta \pm \sqrt{\left(\frac{16}{5}\beta\right)^2 - 4 \cdot 16}}{2}$$

$$\left(\frac{16}{5}\beta\right)^2 - 16 \cdot 4$$

$$16\left(\frac{16}{25}\beta^2 - 4\right) = 64\left(\frac{4}{25}\beta^2 - 1\right)$$

a) overdamping would require solution of form

$c_1 e^{m_1 t} + c_2 e^{m_2 t}$, $m_1$ & $m_2$ both real

$$\Rightarrow \frac{4}{25}\beta^2 - 1 > 0$$

$$\beta^2 > \frac{25}{4}$$

$$\beta > \frac{5}{2} \qquad (\beta > 0)$$

b) **Critical damping** would imply a solution of form

$$X = c_1 e^{mt} + c_2 t e^{mt} \quad \text{(Roots equal)}$$
$$m_1 = m_2$$

$$\frac{4}{25}\beta^2 - 1 = 0$$
$$\beta^2 = \frac{25}{4}$$
$$\beta = \frac{5}{2}$$

c) **Underdamped**: would require a solution of the form $X = e^{mt}(\cos Bt + \sin Bt)$

$$\frac{4}{25}B^2 - 1 < 0$$
$$\frac{4}{25}B^2 < 1$$
$$B^2 < \frac{25}{4}$$
$$B < \frac{5}{2}$$

5.2 #15 $m = 40g$ stretches spring 10 cm

damping force equals $560x'$

$x(0) = 0, \quad x'(0) = 2\ cm/s$

$g = 980\ cm/s^2$

$W = mg = 40g \cdot 980\ cm/s^2$

$W = 39200$

$W = F = 10k$

$39200 = 10 \cdot k$

$3920 = k$

$mx'' + \beta x' + kx = 0$

$40x'' + 560x' + 3920x = 0$

$x'' + 14x' + 98x = 0$

$$m = \frac{-14 \pm \sqrt{14^2 - 4 \cdot 98}}{2}$$

$$= \frac{-14 \pm \sqrt{-196}}{2}$$

$$= \frac{-14 \pm 14i}{2}$$

$$= -7 \pm 7i$$

$x = e^{-7t}(C_1 \cos 7t + C_2 \sin 7t)$

$x(0) = 0 = C_1 \Rightarrow x = C_2 e^{-7t} \sin 7t$

$x' = -7C_2 e^{-7t} \sin 7t + 7C_2 e^{-7t} \cos 7t$

$x'(0) = 2 = 7C_2 \Rightarrow C_2 = \frac{2}{7}$

$x = \frac{2}{7} e^{-7t} \sin 7t$

5.2 #17 $m = 1$ slug $k = 9$ lb/ft

Damping force of $6x'$

$x(0) = -8$ in $= -\frac{2}{3}$ ft

$x'(0) = v_0$ ft/s

$mx'' + \beta x' + kx = 0$

$x'' + 6x' + 9x = 0$

$(m+3)^2 = 0$

$x = c_1 e^{-3t} + c_2 t e^{-3t} \Rightarrow x' = -3c_1 e^{-3t} + c_2 e^{-3t} - 3c_2 t e^{-3t}$

$x(0) = -\frac{2}{3} = c_1$

$x'(0) = v_0 = -3c_1 + c_2$

$c_2 = v_0 + 3c_1 = v_0 + 3\left(-\frac{2}{3}\right) = v_0 - 2$

$x(t) = -\frac{2}{3} e^{-3t} + (v_0 - 2) t e^{-3t}$

$-\frac{2}{3} e^{-3t} (v_0 - 2) t e^{-3t} = 0$

$\frac{2}{3} = (v_0 - 2) t \Rightarrow \frac{2}{3(v_0 - 2)} = t$

For $t$ to be defined or positive,

$v_0 > 2$