Zill 5th

Math 205 –

7.1 – 7.3 Selected Solutions

7.1 #1 $\mathcal{L}\{f(t)\}$, $f(t) = \begin{cases} -1 & 0 \le t < 1 \\ 1 & t \ge 1 \end{cases}$

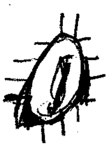
$$\mathcal{L}\{f(t)\} = \int_0^1 e^{-st}(-1)\,dt + \int_1^\infty e^{-st}\,dt$$

$$= \frac{1}{s}e^{-st}\Big|_0^1 - \frac{1}{s}e^{-st}\Big|_1^\infty$$

$$= \frac{e^{-s}}{s} - \frac{1}{s} - \left[0 - \frac{e^{-s}}{s}\right]$$

$$= \frac{2e^{-s}}{s} - \frac{1}{s} \qquad s > 0$$

7.1 #3



$$f(t) = \begin{cases} t & 0 \le t < 1 \\ 1 & t \ge 1 \end{cases}$$

$$\mathcal{L}\{f(t)\} = \int_0^1 te^{-st}\,dt + \int_1^\infty e^{-st}\,dt$$

$$= \left[-\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\right]_0^1 - \frac{1}{s}e^{-st}\Big|_1^\infty$$

$$= -\frac{1}{s}e^{-s} - \frac{1}{s^2}e^{-s} + \frac{1}{s^2} + \frac{e^{-s}}{s}$$

$$= \frac{1}{s^2}(1 - e^{-s}) \qquad s > 0$$

7.1#5 $\quad f(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ 0 & t \ge \pi \end{cases}$

$$\mathcal{L}\{f(t)\} = \int_0^{\pi} e^{-st} \sin t \, dt + \int_{\pi} 0 \, dt$$

$$= \frac{1}{s^2} e^{-s\pi} + \frac{1}{s^2} - \frac{1}{s^2}\int_0^{\pi} \sin t \, e^{-st} \, dt$$

$u = \sin t \quad dv = e^{-st} dt$

$du = \cos t \, dt \quad v = \frac{-e^{-st}}{s}$

$$-\frac{\sin t}{s} e^{-st}\Big|_0^{\pi} + \int_0^{\pi} \frac{e^{-st}}{s} \cos t \, dt$$

$$0 + \frac{1}{s}\int_0^{\pi} e^{-st} \cos t \, dt$$

$u = \cos t \quad dv = e^{-st} dt$

$du = -\sin t \, dt \quad v = -\frac{1}{s} e^{-st}$

$$-\frac{\cos t}{s^2} e^{-st}\Big|_0^{\pi} - \frac{1}{s^2}\int_0^{\pi} \sin t \, e^{-st} \, dt$$

$$\frac{1}{s^2} e^{-s\pi} + \frac{1}{s^2} - \frac{1}{s^2}\int_0^{\pi} e^{-st} \sin t \, dt$$

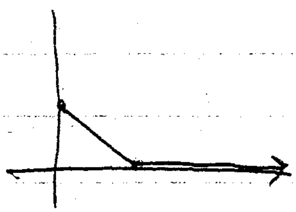
$$\left(1 + \frac{1}{s^2}\right)\int_0^{\pi} e^{-st} \sin t \, dt = \frac{1}{s^2}\left(e^{-s\pi} + 1\right)$$

$$\int_0^{\pi} e^{-st} \sin t \, dt = \frac{\frac{1}{s^2}}{1 + \frac{1}{s^2}}\left(e^{-s\pi} + 1\right)$$

$$= \left(\frac{1}{s^2+1}\right)\left(e^{-s\pi} + 1\right)$$

$$= \frac{e^{-s\pi} + 1}{s^2 + 1}$$

7.1#9



$$\Rightarrow \quad f(t) = \begin{cases} 1 - t & 0 \le t < 1 \\ 0 & x \ge 1 \end{cases}$$

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) \, dt = \int_0^1 (1-t) e^{-st} + \int_1^{\infty} 0 \, e^{-st} \, dt$$

$$= \int_0^1 e^{-st} dt - \int_0^1 t e^{-st} dt \longrightarrow$$

$u = t \quad dv = e^{-st} dt$

$du = dt \quad v = -\frac{1}{s} e^{-st}$

$$= -\frac{1}{s} e^{-st}\Big|_0^1 - \left[-\frac{t}{s} e^{-st} - \frac{1}{s^2} e^{-st}\right]_0^1$$

$$= -\frac{e^{-s}}{s} + \frac{1}{s} - \left[-\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}\right] = \frac{e^{-s}}{s} + \frac{1}{s} - \frac{1}{s^2}$$

7.1 #11 $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} \cdot e^{t+7}\, dt \qquad f(t) = e^{t+7}$

$= \int_0^\infty e^{7+(1-s)t}\, dt$

$= \left. \dfrac{e^{7+(1-s)t}}{1-s} \right|_0^\infty$

$= 0 - \dfrac{e^7}{1-s}$

$= \dfrac{e^7}{s-1} \qquad s > 1$

7.1 #15 $f(t) = e^{-t} \sin t$

$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-t} \sin t\, e^{-st}\, dt$

$= \int_0^\infty e^{-t(s+1)} \sin t\, dt$

$\mathcal{L}\{f(t)\} = \dfrac{1}{(s+1)^2} - \dfrac{1}{(s+1)^2}\mathcal{L}\{f(t)\}$ $\qquad u = \sin t \quad dv = e^{-(s+1)t}\, dt$

$du = \cos t\, dt \quad v = -\dfrac{e^{-(s+1)t}}{(s+1)}$

$\left[1 + \dfrac{1}{(s+1)^2}\right]\mathcal{L}\{f(t)\} = \dfrac{1}{(s+1)^2}$ $\qquad \left.\dfrac{-\sin t\, e^{-(s+1)t}}{s+1}\right|_0^\infty + \dfrac{1}{s+1}\int_0^\infty \cos t\, e^{-(s+1)t}\, dt$

$\mathcal{L}\{f(t)\} = \dfrac{\frac{1}{(s+1)^2}}{1 + \frac{1}{(s+1)^2}}$ $\qquad u = \cos t \quad dv = e^{-(s+1)t}\, dt$

$du = \sin t\, dt \quad v = \dfrac{e^{-(s+1)t}}{-(s+1)}$

$= \dfrac{1}{(s+1)^2 + 1}$ $\qquad \left.-\dfrac{\cos t}{(s+1)^2} e^{-(s+1)t}\right|_0^\infty - \dfrac{1}{s+1}\int_0^\infty \sin t\, e^{-(s+1)t}\, dt$

$= \dfrac{1}{s^2 + 2s + 2}$ $\qquad \dfrac{1}{(s+1)^2} - \dfrac{1}{(s+1)^2}\mathcal{L}\{f(t)\}$

7.1#19 $\mathcal{L}\{2t^4\} = \dfrac{2\cdot 4!}{s^5}$

7.1#21 $\mathcal{L}\{4t-10\} = \dfrac{4}{s^2} - \dfrac{10}{s}$

7.1#23 $\mathcal{L}\{t^2+6t-3\} = \dfrac{2}{s^3} + \dfrac{6}{s^2} - \dfrac{3}{s}$

7.1#25 $\mathcal{L}\{(t+1)^3\} = \mathcal{L}\{t^3+3t^2+3t+1\} = \dfrac{3!}{s^4} + \dfrac{3\cdot 2}{s^3} + \dfrac{3}{s^2} + \dfrac{1}{s}$

7.1#27 $\mathcal{L}\{1+e^{4t}\} = \dfrac{1}{s} + \dfrac{1}{s-4}$

7.1#29 $\mathcal{L}\{(1+e^{2t})^2\} = \mathcal{L}\{1+2e^{2t}+e^{4t}\}$

$= \dfrac{1}{s} + \dfrac{2}{s-2} + \dfrac{1}{s-4}$

7.1#31 $\mathcal{L}\{4t^2-5\sin 3t\} = \dfrac{4\cdot 2}{s^3} - \dfrac{5\cdot 3}{s^2+9}$

7.1#33 $\mathcal{L}\{\sinh kt\} = \dfrac{k}{s^2-k^2}$

7.1#35 $\mathcal{L}\{e^t \sinh t\} = \frac{1}{2}\mathcal{L}\{e^t(e^t-e^{-t})\}$

$= \frac{1}{2}\mathcal{L}\{e^{2t}-1\}$

$= \frac{1}{2}\cdot\dfrac{1}{s-2} - \dfrac{1}{2s}$

7.1#37 $\mathcal{L}\{\sin 2t\cos 2t\}$

$= \mathcal{L}\{\frac{1}{2}\sin 4t\}$

$\sin 2t = 2\sin t\cos t$

$\sin 4t = 2\sin 2t\cos 2t$

$= \dfrac{1}{2}\,\dfrac{4}{s^2+16}$

$= \dfrac{2}{s^2+16}$

7.1 #39 $\mathcal{L}\{\cos t \cos 2t\}$

$$\cos(t+2t) = \cos t \cos 2t - \sin t \sin 2t$$

$$\cos 3t = \cos t \cos 2t - \sin t \sin 2t$$

$$\cos(t-2t) = \cos t \cos 2t + \sin t \sin 2t$$

$$\cos 3t + \cos(-t) = 2\cos t \cos 2t$$

$$\frac{\cos 3t + \cos t}{2} = \cos t \cos 2t$$

$$\mathcal{L}\{\cos t \cos 2t\} = \mathcal{L}\left\{\frac{\cos 3t + \cos t}{2}\right\}$$

$$= \frac{1}{2} \cdot \frac{s}{s^2+9} + \frac{1}{2} \cdot \frac{s}{s^2+1}$$

7.1 #41 $\mathcal{L}\{\cos 2t \sin t\}$

$$\sin 3t = \sin t \cos 2t + \cos t \sin 2t$$

$$\sin(-t) = \sin t \cos 2t - \cos t \sin 2t$$

$$\sin 3t - \sin t = 2\sin t \cos 2t$$

$$\frac{\sin 3t - \sin t}{2} = \sin t \cos 2t$$

$$\mathcal{L}\{\sin t \cos 2t\} = \mathcal{L}\left\{\frac{\sin 3t - \sin t}{2}\right\}$$

$$= \frac{1}{2} \frac{3}{s^2+9} - \frac{1}{2} \frac{1}{s^2+1}$$

7.1 #43 $\mathcal{L}\{t^\alpha\} = \int_0^\infty e^{-st} t^\alpha \, dt$

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt \quad , \alpha > 0$$

Show $\mathcal{L}\{t^\alpha\} = \dfrac{\Gamma(\alpha+1)}{s^{\alpha+1}}, \quad \alpha > -1$

using $w = st$,

$$\int_0^\infty e^{-st} t^\alpha dt = \int_0^\infty e^{-w} \left(\frac{w}{s}\right)^\alpha \frac{dw}{s}$$

$$= \int_0^\infty e^{-w} \frac{w^\alpha}{s^{\alpha+1}} dw = \frac{1}{s^{\alpha+1}} \int_0^\infty e^{-w} w^\alpha dw$$

$$= \frac{1}{s^{\alpha+1}} \Gamma(\alpha+1)$$

$$\mathcal{L}^{-1}\{\} = \mathcal{L}^{-1}\left\{\frac{1/2}{s} - \frac{1}{s-1} - \frac{1/3}{s+1} + \frac{5/6}{s-2}\right\}$$

$$= \frac{1}{2} - e^{t} - \frac{1}{3}e^{-t} + \frac{5}{6}e^{2t}$$

7.2 #1 $\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{1}{2}t^2$

7.2 #5 $\mathcal{L}^{-1}\left\{\frac{(s+1)^3}{s^4}\right\} = \mathcal{L}^{-1}\left\{\frac{s^3+3s^2+3s+1}{s^4}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{3}{s^2} + \frac{3}{s^3} + \frac{1}{s^4}\right\}$$

$$= 1 + 3t + \frac{3}{2}t^2 + \frac{1}{6}t^3$$

7.2 #11 $\mathcal{L}^{-1}\left\{\frac{5}{s^2+49}\right\} = \frac{5}{7}\sin 7t$

7.2 #13 $\mathcal{L}^{-1}\left\{\frac{4s}{4s^2+1}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1/4}\right\}$

$$= \cos\frac{1}{2}t$$

7.2 #15 $\mathcal{L}^{-1}\left\{\frac{1}{s^2-16}\right\} = \frac{\sinh 4t}{4}$

7.2 #17 $\mathcal{L}^{-1}\left\{\frac{2s-6}{s^2+9}\right\} = \mathcal{L}^{-1}\left\{\frac{2s}{s^2+9}\right\} - 6\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}$

$$= 2\cos 3t - 2\sin 3t$$

7.2 # 21 $\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s-3}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{(s+3)(s-1)}\right\}$

$$\frac{s}{(s+3)(s-1)} = \frac{A}{s+3} + \frac{B}{s-1}$$

$$s = A(s-1) + B(s+3)$$

$s=1 \Rightarrow \quad 1 = 4B$

$\frac{1}{4} = B$

$s=-3 \Rightarrow \quad -3 = -4A$

$3/4 = A$

$$= \mathcal{L}^{-1}\left\{\frac{3/4}{s+3} + \frac{1/4}{s-1}\right\}$$

$$= \frac{3}{4}e^{-3t} + \frac{1}{4}e^{t}$$

7.2 # 23 $\mathcal{L}^{-1}\left\{\frac{0.9s}{(s-.1)(s+.2)}\right\} = \mathcal{L}^{-1}\left\{\frac{A}{s-.1}\right\} + \mathcal{L}^{-1}\left\{\frac{B}{s+.2}\right\}$

$$\frac{.9s}{(s-.1)(s-.2)} = \frac{A}{s-.1} + \frac{B}{s+.2}$$

$$.9s = A(s+.2) + B(s-.1)$$

$s=.1 \Rightarrow \quad .09 = .3A$ $\qquad$ $s=-.2 \Rightarrow \quad -.18 = -.3B$

$\frac{.09}{.3} = A$ $\qquad$ $\frac{-.18}{-.3} = B$

$.3 = A$ $\qquad$ $+0.6 = B$

$$= \mathcal{L}^{-1}\left\{\frac{.3}{s-.1}\right\} + \mathcal{L}^{-1}\left\{\frac{.6}{s-.2}\right\}$$

$$= .3e^{.1t} + .6e^{.2t}$$

7.2 # 21 $\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s-3}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{(s+3)(s-1)}\right\}$

$$\frac{s}{(s+3)(s-1)} = \frac{A}{s+3} + \frac{B}{s-1}$$

$$s = A(s-1) + B(s+3)$$

$s=1 \Rightarrow \quad 1 = 4B$

$\frac{1}{4} = B$

$s=-3 \Rightarrow \quad -3 = -4A$

$3/4 = A$

$$= \mathcal{L}^{-1}\left\{\frac{3/4}{s+3} + \frac{1/4}{s-1}\right\}$$

$$= \frac{3}{4}e^{-3t} + \frac{1}{4}e^{t}$$

7.2 # 23 $\mathcal{L}^{-1}\left\{\frac{0.9s}{(s-.1)(s+.2)}\right\} = \mathcal{L}^{-1}\left\{\frac{A}{s-.1}\right\} + \mathcal{L}^{-1}\left\{\frac{B}{s+.2}\right\}$

$$\frac{.9s}{(s-.1)(s-.2)} = \frac{A}{s-.1} + \frac{B}{s+.2}$$

$$.9s = A(s+.2) + B(s-.1)$$

$s=.1 \Rightarrow \quad .09 = .3A$ $\qquad s=-.2 \Rightarrow \quad -.18 = -.3B$

$\frac{.09}{.3} = A$ $\qquad \frac{-.18}{-.3} = B$

$.3 = A$ $\qquad +0.6 = B$

$$= \mathcal{L}^{-1}\left\{\frac{.3}{s-.1}\right\} + \mathcal{L}^{-1}\left\{\frac{.6}{s-.2}\right\}$$

$$= .3e^{.1t} + .6e^{.2t}$$

7.2 # 27 $\mathcal{L}^{-1}\left\{\frac{2s+4}{(s-2)(s^2+4s+3)}\right\}$

$$\frac{A}{s-2}+\frac{B}{s+3}+\frac{C}{s+1}=\frac{2s+4}{(s-2)(s+3)(s+1)}$$

$$A(s+3)(s+1)+B(s-2)(s+1)+C(s-2)(s+3)=2s+4$$

$s=-1 \Rightarrow \quad -6C=2$

$C=-\frac{1}{3}$

$s=2 \Rightarrow \quad 15A=8$

$A=\frac{8}{15}$

$s=-3 \Rightarrow \quad 6B=-2$

$B=-\frac{1}{3}$

$$=\mathcal{L}^{-1}\left\{\frac{\frac{8}{15}}{s-2}-\frac{\frac{1}{3}}{s+3}-\frac{\frac{1}{3}}{s+1}\right\}$$

$$=\frac{8}{5}e^{2t}-\frac{1}{3}e^{-3t}-\frac{1}{3}e^{-t}$$

7.2 # 29 $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\}=\mathcal{L}^{-1}\left\{\frac{A}{s}+\frac{B}{s^2}+\frac{Cs+D}{s^2+4}\right\}$

$$As(s^2+4)+B(s^2+4)+(Cs+D)s^2=1$$

$s=0 \Rightarrow \quad 4B=1$

$B=\frac{1}{4}$

$$As^3+4As+Bs^2+4B+Cs^3+Ds^2=1$$

$A+C=0, \quad B+D=0, \quad 4A=0$

$A=0 \Rightarrow C=0$

$-\frac{1}{4}=D$