Mth 205 - Monbrod

Solutions - Selected Exercises 7.4, 7.5, Zill 5th

7.4 # 1. if $\frac{d}{dt}(e^t) = e^t$, $\mathcal{L}\{e^t\} = ?$ by $\mathcal{L}\{f'\} = sF(s) - f(0)$

$$\mathcal{L}\{f'\} = sF(s) - f(0)$$

$$\mathcal{L}\{e^t\} = s\mathcal{L}\{e^t\} - e^0$$

$$\mathcal{L}\{e^t\} - s\mathcal{L}\{e^t\} = -1$$

$$\mathcal{L}\{e^t\} = \frac{-1}{1-s}$$

$$= \frac{1}{s-1}$$

7.4 # 3 $y(0) = 1$, $y'(0) = -1$. Find $\mathcal{L}\{y'' + 3y'\}$

$$\mathcal{L}\{y'' + 3y'\}$$

$$= s^2 Y(s) - sy(0) - y'(0) + 3sY(s) - 3y(0)$$

$$= (s^2 + 3s)Y(s) - s + 1 - 3$$

$$= (s^2 + 3s)Y(s) - s - 2$$

7.4 # 5 $y(0) = 2$, $y'(0) = 3$, Find $Y(s)$

$$y'' - 2y' + y = 0$$

$$s^2 Y(s) - sy(0) - y'(0) - 2sY(s) + 2y(0) + Y(s) = 0$$

$$Y(s)(s^2 - 2s + 1) - 2s - 3 + 4 = 0$$

$$Y(s) = \frac{2s - 1}{s^2 - 2s + 1}$$

7-19 Find without evaluating integral

7.4 #7 $\mathcal{L}\left\{\int_0^t e^{\tau}\,d\tau\right\} = \mathcal{L}\left\{\int_0^t 1\cdot e^{\tau}\,d\tau\right\} = \mathcal{L}\{1 * e$

$$= \mathcal{L}\{1\}\,\mathcal{L}\{e^t\}$$

$$= \frac{1}{s}\cdot\frac{1}{s-1}$$

7.4 #9 $\mathcal{L}\left\{\int_0^t e^{-\tau}\cos\tau\,d\tau\right\} = \mathcal{L}\{1 * e^{-t}\cos t\}$

$$= \mathcal{L}\{1\}\,\mathcal{L}\{e^{-t}\cos t\}$$

$$= \frac{1}{s}\cdot\frac{s+1}{(s+1)^2+1}$$

7.4 #11 $\mathcal{L}\left\{\int_0^t \tau e^{t-\tau}\,d\tau\right\} = \mathcal{L}\left[t * e^t\right]$

$$= \mathcal{L}\{t\}\,\mathcal{L}\{e^t\}$$

$$= \frac{1}{s^2}\cdot\frac{1}{s-1}$$

7.4 #13 $\mathcal{L}\left\{t\int_0^t \sin\tau\,d\tau\right\} = -\frac{d}{ds}\left[\mathcal{L}\left\{\int_0^t \sin\tau\,d\tau\right\}\right]$

$$= -\frac{d}{ds}\left[\mathcal{L}\{1 * \sin t\}\right]$$

$$= -\frac{d}{ds}\left[\frac{1}{s}\cdot\frac{1}{s^2+1}\right]$$

$$= -\frac{d}{ds}\left[\frac{1}{s(s^2+1)}\right]$$

$$= -\left[\frac{-(s^2+1)-2s^2}{s^2(s^2+1)^2}\right]$$

$$= \frac{3s^2+1}{s^2(s^2+1)^2}$$

7.4 #15 $\mathcal{L}\{1 * t^3\} = \mathcal{L}\{1\} \cdot \mathcal{L}\{t^3\}$

$$= \frac{1}{s} \cdot \frac{3!}{s^4}$$

$$= \frac{6}{s^5}$$

7.4 #17 $\mathcal{L}\{t^2 * t^4\} = \mathcal{L}\{t^2\}\mathcal{L}\{t^4\}$

$$= \frac{2}{s^3} \cdot \frac{4!}{s^5}$$

$$= \frac{48}{s^8}$$

7.4 #19

$$\mathcal{L}\{e^{-t} * e^t \cos t\} = \mathcal{L}\{e^{-t}\}\mathcal{L}\{e^t \cos t\}$$

$$= \frac{1}{s+1} \cdot \frac{s-1}{(s-1)^2+1}$$

$$= \frac{s-1}{(s+1)((s-1)^2+1)}$$

7.4 #21

$\frac{1}{s+5} \cdot F(s)$ Assume $\mathcal{L}^{-1}\{F(s)\} = f(t)$

$$\mathcal{L}^{-1}\left\{\frac{1}{s+5} \cdot F(s)\right\} = e^{-5t} * f(t)$$

$$= \int_0^t e^{-5\tau} f(t-\tau)\, d\tau$$

$$= \int_0^t f(\tau)\, e^{-5(t-\tau)}\, d\tau$$

7.~~4~~ #23 $\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\}$ use convolution

$$= 1 * e^{-t}$$

$$= \int_0^t e^{-\tau}\, d\tau$$

$$= -e^{-\tau}\Big|_0^t$$

$$= -e^{-t} + 1$$

$$= 1 - e^{-t}$$

7.~~4~~ #25 $\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s-2)}\right\}$

$$= e^{-t} * e^{2t}$$

$$= \int_0^t e^{-\tau} e^{2(t-\tau)}\, d\tau$$

$$= \int_0^t e^{2t-3\tau}\, d\tau$$

$$= -\frac{1}{3} e^{2t-3\tau}\Big|_0^t$$

$$= -\frac{1}{3}\left[e^{2t-3t} - e^{2t-0}\right]$$

$$= -\frac{1}{3} e^{-t} + \frac{1}{3} e^{2t}$$

7-#27 $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)^2}\right\}$

$$= \frac{\sin 2t}{2} * \cos 2t$$

$$= \int_0^t \frac{\sin 2\tau}{2}(\cos 2(t-\tau))\,d\tau$$

$$= \int_0^t \frac{\sin 2\tau}{2}(\cos 2t \cos 2\tau + \sin 2t \sin 2\tau)\,d\tau$$

$$= \frac{1}{2}\int_0^t [\cos 2t \sin 2\tau \cos 2\tau + \sin 2t \sin^2 2\tau]\,d\tau$$

$$= \frac{1}{2}\int_0^t \left[\frac{\cos 2t \sin 4\tau}{2} + \frac{\sin 2t}{2}(1-\cos 4\tau)\right]d\tau$$

$$= \frac{1}{2}\left[\frac{\cos 2t}{2}\left(\frac{-1}{4}\cos 4\tau\right) + \frac{\sin 2t}{2}\left(\tau - \frac{1}{4}\sin 4\tau\right)\right]\Bigg|_0^t$$

$$= \frac{1}{2}\left[-\frac{1}{8}\cos 2t \cos 4t + \frac{\sin 2t}{2}\left(t - \frac{1}{4}\sin 4t\right) + \frac{\cos 2t}{8} - 0\right]$$

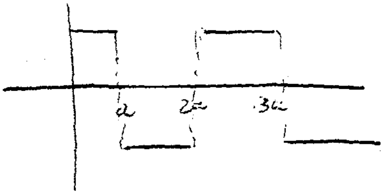
$$= \frac{1}{2}\left[-\frac{1}{8}\cos 2t \cos 4t - \frac{1}{8}\sin 2t \sin 4t + \frac{t}{2}\sin 2t + \frac{\cos 2t}{8}\right]$$

$$= \frac{1}{2}\left[-\frac{1}{8}\cos(-2t) + \frac{t}{2}\sin 2t + \frac{\cos 2t}{8}\right]$$

$$= \frac{1}{2}\left[\frac{t}{2}\sin 2t - \frac{1}{8}\cos 2t + \frac{\cos 2t}{8}\right]$$

$$= \frac{t \sin 2t}{4}$$

7.4 # 31



Period = 2a

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-2as}} \int_0^{2a} f(t) e^{-st} \, dt$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^{a} e^{-st} \, dt + \int_a^{2a} -e^{-st} \, dt \right]$$

$$= \frac{1}{1-e^{-2as}} \left[-\frac{1}{s} e^{-st} \Big|_0^{a} + \frac{e^{-st}}{s} \Big|_a^{2a} \right]$$

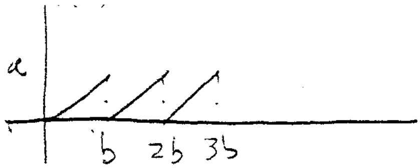
$$= \frac{1}{1-e^{-2as}} \left[-\frac{1}{s} e^{-as} + \frac{1}{s} + \frac{e^{-2as}}{s} - \frac{e^{-as}}{s} \right]$$

$$= \frac{1}{s} \cdot \left(\frac{1 - 2e^{-as} + e^{-2as}}{1 - e^{-2as}} \right)$$

$$= \frac{(1 - e^{-as})^2}{s(1 - e^{-as})(1 + e^{-as})}$$

$$= \frac{(1 - e^{-as})}{s(1 + e^{-as})}$$

7.4 #33



$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-bs}} \int_0^b \frac{a}{b} t\, e^{-st}\, dt$$

$u = t \quad dv = e^{-st}\,dt$

$du = dt \quad v = -\frac{1}{s}e^{-st}$

$$= \frac{a/b}{1-e^{-bs}} \left[-\frac{t}{s} e^{-st} \Big|_0^b + \frac{1}{s} \int_0^b e^{-st}\, dt \right]$$

$$= \frac{a/b}{1-e^{-bs}} \left[\frac{-b e^{-bs}}{s} - \frac{1}{s^2} e^{-st} \Big|_0^b \right]$$

$$= \frac{a/b}{1-e^{-bs}} \left[-\frac{b e^{-bs}}{s} - \frac{e^{-bs}}{s^2} + \frac{1}{s^2} \right]$$

$$= \frac{a}{1-e^{-bs}} \left[-\frac{e^{-bs}}{s} - \frac{e^{-bs}}{bs^2} + \frac{1}{bs^2} \right]$$

$$= \frac{a}{s} \left(\frac{-e^{-bs}}{1-e^{-bs}} \right) + \frac{a(1-e^{-bs})}{(1-e^{-bs})bs^2}$$

$$= \frac{a}{s} \left(\frac{-e^{-bs} e^{bs}}{e^{bs}-1} \right) + \frac{a}{bs^2}$$

$$= \frac{a}{s} \left(\frac{1}{1-e^{bs}} \right) + \frac{a}{bs^2}$$

$$= \frac{a}{s} \left(\frac{1}{bs} - \frac{1}{e^{bs}-1} \right)$$