7.5# 5 $y'' + 5y' + 4y = 0, \quad y(0) = 1, \; y'(0) = 0$

$$s^2 Y(s) - s y(0) - y'(0) + 5s Y(s) - 5y(0) + 4 Y(s) = 0$$

$$(s^2 + 5s + 4) Y(s) - s - 5 = 0$$

$$Y(s) = \frac{s+5}{s^2+5s+4}$$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{s+5}{(s+1)(s+4)} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{4/3}{s+1} - \frac{1/3}{s+4} \right\}$$

$$= \frac{4}{3} e^{-t} - \frac{1}{3} e^{-4t}$$

$$\frac{A}{s+1} + \frac{B}{s+4} = \frac{s+5}{(s+1)(s+4)}$$

$$A(s+4) + B(s+1) = s+5$$

$$As + 4A + Bs + B = s + 5$$

$$A + B = 1 \quad 4A + B = 5$$

$$3A = 4$$

$$A = 4/3$$

$$B = -1/3$$

7.5# 11 $y'' + y = \sin t, \quad y(0) = 1, \; y'(0) = 1$

$$s^2 Y(s) - s y(0) - y'(0) + Y(s) = \frac{1}{s^2+1}$$

$$(s^2+1) Y(s) - s + 1 = \frac{1}{s^2+1}$$

$$(s^2+1) Y(s) = \frac{1}{s^2+1} + s - 1$$

$$Y(s) = \frac{1}{(s^2+1)^2} + \frac{s-1}{s^2+1}$$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{1}{(s^2+1)^2} + \frac{s-1}{s^2+1} \right\}$$

$$= \mathcal{L}\left\{ \frac{1}{(s^2+1)^2} \right\} + \mathcal{L}^{-1}\left\{ \frac{s}{s^2+1} - \frac{1}{s^2+1} \right\}$$

See details Next Page

$$= \mathcal{L}^{-1}\left\{ \frac{1}{s^2+1} \cdot \frac{1}{s^2+1} \right\} + \cos t - \sin t$$

$$= \sin t * \sin t + \cos t - \sin t$$

$$= \frac{\sin t}{2} - \frac{t}{2} \cos t + \cos t - \sin t = \cos t - \frac{t}{2} \cos t - \frac{1}{2} \sin$$

$\sin t * \sin t$

$$= \int_0^t \sin\tau \, \sin(t-\tau)\, d\tau$$

$$= \int_0^t \sin\tau \left(\sin t \cos\tau - \cos t \sin\tau \right) d\tau$$

$$= \int_0^t \left(\sin t \sin\tau\cos\tau - \cos t \sin^2\tau \right) d\tau$$

$$= \sin t \left. \frac{\sin^2\tau}{2} \right|_0^t - \cos t \int_0^t \frac{1-\cos 2\tau}{2}\, d\tau$$

$$= \frac{\sin^3 t}{2} - \cos t \left[\frac{\tau}{2} - \frac{\sin 2\tau}{4} \right]_0^t$$

$$= \frac{\sin^3 t}{2} - \cos t \left[\frac{t}{2} - \frac{\sin 2t}{4} \right]$$

$$= \frac{\sin^3 t}{2} - \frac{t}{2}\cos t + \frac{\cos t}{4}\, 2\sin t \cos t$$

$$= \frac{\sin^3 t}{2} - \frac{t}{2}\cos t + \frac{\cos^2 t \sin t}{2}$$

$$= \frac{\sin^3 t}{2} - \frac{t}{2}\cos t + \frac{(1-\sin^2 t)\sin t}{2}$$

$$= \frac{\sin^3 t}{2} - \frac{t}{2}\cos t + \frac{1}{2}\sin t - \frac{1}{2}\sin^3 t$$

$$= -\frac{t}{2}\cos t + \frac{1}{2}\sin t$$

See Next Page For Alternate approach to finding $\mathcal{L}^{-1}\left\{ \frac{1}{(s^2+1)^2} \right\}$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{2s}\cdot(-1)\frac{d}{ds}\left[\frac{1}{s^2+1}\right]\right\}$$

$$= \frac{1}{2} * t\sin t$$

By Parts:

$u = \tau \qquad dv = \sin\tau\, d\tau$

$du = d\tau \qquad v = -\cos\tau$

$$= \frac{1}{2}\int_0^t \tau\sin\tau\, d\tau$$

$$= \frac{1}{2}\left[-\tau\cos\tau\Big|_0^t + \int_0^t \cos\tau\, d\tau\right]$$

$$= \frac{1}{2}\left[-t\cos t + \sin t\right]$$

$$= \frac{1}{2}\sin t - \frac{t}{2}\cos t$$

Note: Seeking alternate solutions is excellent practice, but the alternate solutions are not always shorter.

7.5#9 $\quad y'' - 4y' + 4y = t^3 e^{2t} \qquad y(0) = 0,\ y'(0) = 0$

$$s^2Y(s) - sy(0) - y'(0) - 4sY(s) + 4y(0) + 4Y(s) = \frac{6}{(s-2)^4}$$

$$(s^2 - 4s + 4)Y(s) - 0 - 0 + 0 = \frac{6}{(s-2)^4}$$

$$Y(s) = \frac{6}{(s-2)^2(s-2)^4}$$

$$Y(s) = \frac{6}{(s-2)^6}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{6}{(s-2)^6}\right\}$$

$$= \frac{6}{5!} t^5 e^{2t}$$

$$= \frac{t^5 e^{2t}}{20}$$

7.5#20 $\quad y' + y = f(t), \quad f(t) = \begin{cases} 1 & 0 \le t < 1 \\ -1 & t \ge 1 \end{cases}$

$$sY(s) - y(0) + Y(s) = \frac{1}{s} - \frac{2e^{-s}}{s} \qquad y(0) = 0$$

$$(s+1)Y(s) = \frac{1}{s} - \frac{2e^{-s}}{s} \qquad \text{note } f(t) = \quad - 2\mathcal{U}(t-1)$$

$$Y(s) = \frac{1}{s(s+1)} - \frac{2e^{-s}}{s(s+1)} = \frac{1-2e^{-s}}{s(s+1)} = 1-2e^{-s}\left(\frac{1+s}{s(s+1)} - \frac{s}{s(s+1)}\right.$$

$$= (1-2e^{-s})\left(\frac{1}{s} - \frac{1}{s+1}\right)$$

$$= \frac{1}{s} - \frac{1}{s+1} - \frac{2e^{-s}}{s} + \frac{2e^{-s}}{s+1}$$

$$= 1 - e^{-t} - 2\mathcal{U}(t-1) + 2\mathcal{U}(t-1)e^{-(t-1)}$$

$$= 1 - e^{-t} - 2(\mathcal{U}(t-1))(1 - e^{-(t-1)})$$

7-5 #13 $y'' - y' = e^t \cos t$, $y(0) = 0$, $y'(0) = 0$

$$s^2 Y(s) - s y(0) - y'(0) - sY(s) + y(0) = \mathcal{L}\{e^t \cos t\}$$

$$(s^2 - s)\, Y(s) = \frac{s-1}{(s-1)^2 + 1}$$

$$Y(s) = \frac{(s-1)}{s(s-1)((s-1)^2+1)}$$

$$= \frac{1}{s((s-1)^2+1)}$$

$$y(t) = 1 * e^t \sin t$$

$$= \int_0^t e^\tau \sin \tau \, d\tau$$

$$= \underline{\frac{1}{2} e^t \sin t - \frac{1}{2} e^t \cos t + \frac{1}{2}}$$

$u = \sin\tau \quad dv = e^\tau d\tau$

$du = \cos\tau\, d\tau \quad v = e^\tau$

$e^\tau \sin\tau \Big|_0^t - \int_0^t e^\tau \cos\tau \, d\tau$

$e^t \sin t - \quad u = \cos\tau \quad dv = e^\tau d\tau$

$du = -\sin\tau\, d\tau \quad v = e^\tau$

$e^\tau \cos\tau \Big|_0^t + \int_0^t e^\tau \sin\tau \, d\tau$

$e^t \sin t - e^t \cos t + 1 + \int_0^t e^\tau \sin\tau \, d\tau$

OR $$Y(s) = \frac{A}{s} + \frac{Bs + C}{(s-1)^2+1}$$

$$= \frac{1/2}{s} + \frac{-\frac{1}{2}s + 1}{(s-1)^2+1}$$

$$= \frac{1}{2s} - \frac{1}{2} \cdot \frac{s-2}{(s-1)^2+1}$$

$$= \frac{1}{2s} - \frac{1}{2}\left[\frac{s-1}{(s-1)^2+1} - \frac{1}{(s-1)^2+1}\right]$$

$$y(t) = \frac{1}{2} - \frac{1}{2} e^t \cos t + \frac{1}{2} e^t \sin t$$

7.5#17 $y^{(4)} - y = 0$, $y(0)=1$, $y'(0)=0$, $y''(0)=-1$, $y'''(0)=0$

$$s^4 Y(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) - Y(s) = 0$$

$$s^4 Y(s) - s^3 + s - Y(s) = 0$$

$$(s^4 - 1) Y(s) = s^3 - s$$

$$Y(s) = \frac{s^3 - s}{s^4 - 1}$$

$$= \frac{s(s^2-1)}{(s^2-1)(s^2+1)}$$

$$= \frac{s}{s^2+1}$$

$$y(t) = \cos t$$

7.5# 19 $y' + y = f(t)$, $f(t) = \begin{cases} 0 & 0 \le t < 1 \\ 5 & t \ge 1 \end{cases}$, $y(0)=0$

$$sY(s) - y(0) + Y(s) = \frac{5e^{-s}}{s}$$

note
$f(t) = 5u(t-1)$

$$(s+1)Y(s) = \frac{5e^{-s}}{s}$$

$$Y(s) = \frac{5e^{-s}}{s(s+1)} = 5e^{-s}\left(\frac{A}{s} + \frac{B}{s+1}\right)$$

$$= 5e^{-s}\left(\frac{1}{s} - \frac{1}{s+1}\right)$$

$A(s+1) + Bs = 1$

$As + A + Bs = 1$

$A + B = 0$, $A = 1$

$B = -1$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{5e^{-s}}{s} - \frac{5e^{-s}}{s+1}\right\}$$

$$= \left(5 - 5e^{-(t-1)}\right) u(t-1)$$

$$= 5u(t-1) - 5e^{-(t-1)} u(t-1)$$

7.5 #25 $\quad y'' + y = f(t), \quad f(t) = \begin{cases} 0 & 0 \le t < \pi \\ 1 & \pi \le t < 2\pi \\ 0 & t \ge 2\pi \end{cases}, \quad y(0)=0, \quad y'(0)=$

$$s^2 Y(s) - s\,y(0) - y'(0) + Y(s) = \mathcal{L}\{f(t)\}$$

note $f(t) = u(t-\pi) - u(t-2\pi)$

$$(s^2+1)Y(s) - 1 = \frac{e^{-\pi s}}{s} - \frac{e^{-2\pi s}}{s}$$

$$Y(s) = \left(1 + \frac{e^{-\pi s}}{s} - \frac{e^{-2\pi s}}{s}\right)\cdot\frac{1}{s^2+1}$$

$$= \frac{1}{s^2+1} + \frac{e^{-\pi s} - e^{-2\pi s}}{s(s^2+1)}$$

$$= \frac{1}{s^2+1} + \left(e^{-\pi s} - e^{-2\pi s}\right)\left[\frac{A}{s} + \frac{Bs+C}{s^2+1}\right]$$

$$= \frac{1}{s^2+1} + \left(e^{-\pi s} - e^{-2\pi s}\right)\left(\frac{1}{s} - \frac{s}{s^2+1}\right)$$

$$\frac{A}{s} + \frac{Bs+c}{s^2+1} = 1$$

$$A(s^2+1) + (Bs+C)s = 1$$

$$As^2 + A + Bs^2 + Cs = 1$$

$$A+B=0, \quad C=0, \quad A=1$$

$$B=-1$$

$$= \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s} - \frac{se^{-\pi s}}{s^2+1} - \frac{e^{-2\pi s}}{s} + \frac{se^{-2\pi s}}{s^2+1}$$

$$= \sin t + u(t-\pi) - u(t-\pi)\cos(t-\pi)$$

$$- u(t-2\pi) + u(t-2\pi)\cos(t-2\pi)$$

$$= \sin t + u(t-\pi)\left[1 - \cos(t-\pi)\right] - u(t-2\pi)\left[1 - \cos(t-2\pi)\right]$$

OR

$$= \sin t + u(t-\pi)\left[1 + \cos t\right] - u(t-2\pi)\left[1 - \cos t\right]$$

7.5 # 29

$$f(t) + \int_0^t (t-\tau) f(\tau)\, d\tau = t$$

$$F(s) + \mathcal{L}\left\{\int_0^t (t-\tau) f(\tau)\, d\tau\right\} = \frac{1}{s^2}$$

$$F(s) + \mathcal{L}\{t * f(t)\} = \frac{1}{s^2}$$

$$F(s) + \frac{1}{s^2} \cdot F(s) = \frac{1}{s^2}$$

$$F(s)\left(1 + \frac{1}{s^2}\right) = \frac{1}{s^2}$$

$$F(s)\left(\frac{s^2+1}{s^2}\right) = \frac{1}{s^2}$$

$$F(s) = \frac{1}{s^2+1}$$

$$f(t) = \sin t$$

7.5 # 31

$$f(t) = te^t + \int_0^t \tau f(t-\tau)\, d\tau$$

$$F(s) = \mathcal{L}\{te^t\} + \mathcal{L}\int_0^t \tau \cdot f(t-\tau)\, d\tau$$

$$= \frac{1}{(s-1)^2} + \mathcal{L}\{t\} \cdot \mathcal{L}\{f(t)\}$$

$$F(s) = \frac{1}{(s-1)^2} + \frac{1}{s^2} F(s)$$

$$F(s)\left(1 - \frac{1}{s^2}\right) = \frac{1}{(s-1)^2}$$

$$F(s)\frac{s^2-1}{s^2} = \frac{1}{(s-1)^2}$$

$$F(s) = \frac{s^2}{(s-1)^2(s^2-1)} = \frac{s^2}{(s-1)^3(s+1)}$$

By Partial Fractions $\rightarrow$ $= \frac{1/8}{s-1} + \frac{3/4}{(s-1)^2} + \frac{1/2}{(s-1)^3} - \frac{1/8}{s+1}$

$$= \frac{1}{8}e^t + \frac{3}{4}te^t + \frac{1}{4}t^2 e^t - \frac{1}{8}e^{-t}$$

7.5 #33 $$f(t) + \int_0^t f(\tau)\, d\tau = 1$$

$$F(s) + \mathcal{L}\{1 * f(t)\} = \frac{1}{s}$$

$$F(s) + \frac{1}{s} \cdot F(s) = \frac{1}{s}$$

$$\left(1 + \frac{1}{s}\right) F(s) = \frac{1}{s}$$

$$\left(\frac{s+1}{s}\right) F(s) = \frac{1}{s}$$

$$F(s) = \frac{1}{s+1}$$

$$f(t) = e^{-t}$$

7.5 #53 $w = mg = 4 = 2k$ 4 lb stretches 2 ft.

$32m = 4$ $2 = k$

$m = \frac{1}{8}$

$$\frac{1}{8}x'' + \frac{7}{8}x' + 2x = 0$$

$$x(0) = -1.5 \text{ ft}$$
$$x'(0) = 0$$

$$x'' + 7x' + 16x = 0$$

$$s^2 X(s) - s\,x(0) - x'(0) + 7s\,X(s) - 7x(0) + 16 X(s) = 0$$

$$X(s)\left(s^2 + 7s + 16\right) + 1.5s + 10.5 = 0$$

$$X(s) = \frac{-1.5s - 10.5}{s^2 + 7s + 16}$$

$$= -\frac{1}{2}\left(\frac{3s + 21}{s^2 + 7s + 16}\right)$$

$$= -\frac{3}{2}\left(\frac{s + 7}{s^2 + 7s + \left(\frac{7}{2}\right)^2 + 16 - \left(\frac{7}{2}\right)^2}\right)$$

$$= -\frac{3}{2}\left(\frac{s+7}{(s+\frac{7}{2})^2 + 16-\frac{49}{4}}\right)$$

$$= -\frac{3}{2}\left(\frac{s+\frac{7}{2}+\frac{7}{2}}{(s+\frac{7}{2})^2+\frac{15}{4}}\right)$$

$$= -\frac{3}{2}\left(\frac{s+7/2}{(s+7/2)^2+(\frac{\sqrt{15}}{2})^2}\right) - \frac{3}{2}\;\frac{7/2}{(s+7/2)^2+(\frac{\sqrt{15}}{2})^2}$$

$$= -\frac{3}{2}e^{-\frac{7}{2}t}\cos\frac{\sqrt{15}}{2}t - \frac{21}{2\sqrt{15}}\mathcal{L}^{-1}\left\{\frac{\frac{\sqrt{15}}{2}}{(s+7/2)^2+(\frac{\sqrt{15}}{2})^2}\right\}$$

$$= -\frac{3}{2}e^{-\frac{7}{2}t}\cos\frac{\sqrt{15}}{2}t \;\; -\frac{21\sqrt{15}}{2\cdot 15}e^{-\frac{7}{2}t}\sin\frac{\sqrt{15}}{2}t$$

$$= -\frac{3}{2}e^{-\frac{7}{2}t}\cos\frac{\sqrt{15}}{2}t - \frac{7\sqrt{15}}{10}e^{-\frac{7}{2}t}\sin\frac{\sqrt{15}}{2}t$$

7.5 #59 $\quad ty'' - y' = t^2 \qquad y(0)=0$

$$\mathcal{L}\{ty''\} - \mathcal{L}\{y'\} = \mathcal{L}\{t^2\}$$

$$-\frac{d}{ds}\left[s^2Y(s) - sy(0) - y'(0)\right] - sY(s) + y(0) = \frac{2}{s^3}$$

$$-\frac{d}{ds}\left[s^2 Y(s) - y'(0)\right] - sY(s) = \frac{2}{s^3}$$

$$-\left[2sY(s) + s^2Y'(s)\right] - sY(s) = \frac{2}{s^3}$$

$$-2sY(s) - s^2Y'(s) - sY(s) = \frac{2}{s^3}$$

$$-s^2Y'(s) - 3sY(s) = \frac{2}{s^3}$$

$$s^2Y'(s) + 3sY(s) = \frac{-2}{s^3}$$

$$sY'(s) + 3Y(s) = \frac{-2}{s^4}$$

could solve as Cauchy-Euler or use integrating factor

$$Y'(s) + \frac{3}{s} Y(s) = \frac{-2}{s^5}$$

$$\mu(s) = e^{\int \frac{3}{s} ds} = e^{3\ln s} = s^3$$

$$s^3 Y'(s) + s^3 \cdot \frac{3}{s} Y(s) = -s^3 \cdot \frac{2}{s^5}$$

$$\frac{d}{ds}\left[s^3 Y(s)\right] = \frac{-2}{s^2}$$

$$s^3 Y(s) = \int \frac{-2}{s^2} ds$$

$$s^3 Y(s) = \frac{2}{s} + c$$

$$Y(s) = \frac{2}{s^4} + \frac{c}{s^3}$$

$$y(t) = \frac{2}{3!} t^3 + \frac{c}{2} t^2$$

$$\boxed{y(t) = \frac{t^3}{3} + \frac{c}{2} t^2}$$